

Non-Arbitrage Under Additional Information for Thin Semimartingale Models*

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Abstract

This paper completes the two studies undertaken in [3] and [4], where the authors quantify the impact of a random time on the No-Unbounded-Risk-with-Bounded-Profit concept (called NUPBR hereafter) when the stock price processes are quasi-left-continuous (do not jump on predictable stopping times). Herein, we focus on the NUPBR for semimartingales models that live on thin predictable sets only and the progressive enlargement with a random time. For this flow of information, we explain how far the NUPBR property is affected when one stops the model by an arbitrary random time or when one incorporates fully an honest time into the model. This also generalizes [8] to the case when the jump times are not ordered in anyway. Furthermore, for the current context, we show how to construct explicitly local martingale deflator under the bigger filtration from those of the smaller filtration.

1 Introduction

We consider a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$, where \mathbb{F} is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration \mathbb{F} represents the flow of public information through time. On this basis, we consider an arbitrary but fixed d -dimensional càdlàg semimartingale, S , which represents the discounted price processes of d -stocks, while the riskless asset's price is assumed to be constant. Beside the initial model $(\Omega, \mathcal{G}, \mathbb{F}, P, S)$, we consider a random time τ , i.e. a non-negative \mathcal{G} -measurable random variable. At the practical level, this random time can model the death time, the default time of a firm, or any occurrence time of an event that might affect the market in some way. The main goal of this paper lies in discussing

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whether the new model (S, \mathbb{F}, τ) is arbitrage free or not. To address this question rigourously, we need to specify the non-arbitrage concept adopted herein on the one hand, as arbitrage in continuous time has competing definitions. On the other hand, one need to model the flow of information that catch both the flow \mathbb{F} and the information represented by τ . To this random time, we associate the process D and the filtration \mathbb{G} given by

$$D := I_{[\tau, +\infty[}, \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \quad \mathcal{G}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(D_u, u \leq s)). \quad (1.1)$$

The filtration \mathbb{G} is the smallest right-continuous filtration which contains \mathbb{F} and makes τ a stopping time. In the probabilistic literature, \mathbb{G} is called the progressive enlargement of \mathbb{F} with τ . To define mathematically the non-arbitrage condition, we need to define some notations that will be useful throughout the paper.

1.1 Some General Notations and Definitions

Throughout the paper, \mathbb{H} denotes a filtration satisfying the usual hypotheses and Q a probability measure on the filtered probability space (Ω, \mathbb{H}) . The set of martingales for the filtration \mathbb{H} under Q is denoted by $\mathcal{M}(\mathbb{H}, Q)$. When $Q = P$, we simply denote $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, \mathbb{H} -adapted and integrable processes.

If $\mathcal{C}(\mathbb{H})$ is a class of \mathbb{H} -adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}(\mathbb{H})$ the set of processes X such that there exists a sequence $(T_n)_{n \geq 1}$ of \mathbb{H} -stopping times that increases to $+\infty$ and the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc}(\mathbb{H}) = \mathcal{C}_0(\mathbb{H}) \cap \mathcal{C}_{loc}(\mathbb{H})$.

For a process K with \mathbb{H} -locally integrable variation, we denote by $K^{o, \mathbb{H}}$ its dual optional projection. The dual predictable projection of K (also called the \mathbb{H} -dual predictable projection) is denoted $K^{p, \mathbb{H}}$. For a process X , we denote ${}^{o, \mathbb{H}}X$ (resp. ${}^{p, \mathbb{H}}X$) its optional (resp. predictable) projection with respect to \mathbb{H} .

For an \mathbb{H} -semi-martingale Y , the set $L(Y, \mathbb{H})$ is the set of \mathbb{H} predictable processes integrable w.r.t. Y and for $H \in L(Y, \mathbb{H})$, we denote $H \cdot Y_t := \int_0^t H_s dY_s$.

As usual, for a process X and a random time ϑ , we denote by X^ϑ the stopped process. To distinguish the effect of filtration, we will denote $\langle \cdot, \cdot \rangle^\mathbb{F}$, or $\langle \cdot, \cdot \rangle^\mathbb{G}$ the sharp bracket (predictable covariation process) calculated in the filtration \mathbb{F} or \mathbb{G} , if confusion may rise. We recall that, for general semi-martingales X and Y , the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X, Y]$.

We recall the definition of thin processes/sets for the reader's convenience

Definitions 1.1. A set $A \subset \Omega \times [0, \infty[$ is thin if, for all $\omega \in \Omega$, the set $A(\omega)$ is countable. A process X is called thin if there exists a sequence of random variables ξ_n and an increasing sequence of random times T_n such that $X_t = \sum_{n=1}^{\infty} \xi_n I_{[T_n, \infty[}$. Its paths vary on a thin set only, and hence

$$X = I_{\bigcup_{n=1}^{\infty} [T_n, \infty[} \cdot X = \sum_{n=1}^{\infty} I_{[T_n, \infty[} \cdot X = \sum_{n=1}^{\infty} I_{[T_n, \infty[} \Delta X_{T_n}.$$

1.2 The non-arbitrage concept

We introduce the non-arbitrage notion that will be addressed in this paper.

Definitions 1.2. An \mathbb{H} -semimartingale X satisfies the No-Unbounded-Profit-with-Bounded-Risk condition under (\mathbb{H}, Q) (called $NUPBR(\mathbb{H}, Q)$ hereafter) if for any $T \in (0, +\infty)$ the set

$$\mathcal{K}_T(X, \mathbb{H}) := \left\{ (H \cdot S)_T \mid H \in L(X, \mathbb{H}) \text{ and } H \cdot X \geq -1 \right\}$$

is bounded in probability under Q . When $Q \sim P$, we simply write, with an abuse of language, X satisfies $NUPBR(\mathbb{H})$.

This definition was given in [3], together with the following .

Proposition 1.3. Let X be an \mathbb{H} -semimartingale. Then the following assertions are equivalent.

- (a) X satisfies $NUPBR(\mathbb{H})$.
- (b) There exist a positive \mathbb{H} -local martingale, Y and an \mathbb{H} -predictable process θ satisfying $0 < \theta \leq 1$ and $Y(\theta \cdot X)$ is a local martingale.

For any \mathbb{H} -semimartingale X , the local martingales fulfilling the assertion (b) of Proposition 1.3 are called σ -martingale densities for X . The set of these σ -martingale densities will be denoted throughout the paper by

$$\mathcal{L}(\mathbb{H}, X) := \{Y \in \mathcal{M}_{loc}(\mathbb{H}) \mid Y > 0, \exists \theta \in \mathcal{P}(\mathbb{H}), 0 < \theta \leq 1, Y(\theta \cdot X) \in \mathcal{M}_{loc}(\mathbb{H})\} \quad (1.2)$$

where, as usual, $\mathcal{P}(\mathbb{H})$ stands for the predictable σ -field on $\Omega \times [0, \infty)$ and by abuse of notation $\theta \in \mathcal{P}(\mathbb{H})$ means that θ is $\mathcal{P}(\mathbb{H})$ -measurable. We state, without proof, an obvious lemma.

Lemma 1.4. For any \mathbb{H} -semimartingale X and any $Y \in \mathcal{L}(\mathbb{H}, X)$, one has $p, \mathbb{H}(Y|\Delta X|) < \infty$ and $p, \mathbb{H}(Y\Delta X) = 0$.

Below, we state a result that was proved in [3], and will be frequently used throughout the paper.

Proposition 1.5. Let X be an \mathbb{H} adapted process. Then, the following assertions are equivalent.

- (a) There exists a sequence $(T_n)_{n \geq 1}$ of \mathbb{H} -stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies $NUPBR(\mathbb{H})$ under Q_n .
- (b) X satisfies $NUPBR(\mathbb{H})$.
- (c) There exists an \mathbb{H} -predictable process ϕ , such that $0 < \phi \leq 1$ and $(\phi \cdot X)$ satisfies $NUPBR(\mathbb{H})$.

We end this section with a simple but useful result for predictable process with finite variation.

Lemma 1.6. Let X be an \mathbb{H} -predictable process with finite variation. Then X satisfies $NUPBR(\mathbb{H})$ if and only if $X \equiv X_0$ (i.e. the process X is constant).

1.3 Our Achievements

Given the modeling of the new flow of the information, our main goal becomes whether (S, \mathbb{G}) satisfies the NUPBR or not when S is an \mathbb{F} -semimartingale. Precisely, we characterise the pair of initial market and the random time (S, τ) for which the new market (S, \mathbb{G}) fulfills the NUPBR. This problem was addressed in [3] and [4] for the parts (S^τ, \mathbb{G}) and $(S - S^\tau, \mathbb{G})$ respectively when S is a quasi-left-continuous process. Thus, the case of thin \mathbb{F} -semimartingale with predictable jumps is not covered in these works. The case of discrete time market with finite horizon is presented in [8]. Hence, the main objective of this work lies in deriving results on the NUPBR for thin processes under additional information generated by a random time. It is important to mention that this work complies the other parts towards understanding the effect of extra information on the NUPBR for general semimartingales. This can be

seen by recalling that for an \mathbb{H} -semimartingale, X , we associate a sequence of \mathbb{H} -predictable stopping times $(T_n^X)_{n \geq 1}$ that exhaust the accessible jump times of X , and put $\Gamma_X := \bigcup_{n=1}^{\infty} \llbracket T_n^X \rrbracket$. Then, we can decompose X as follows.

$$X = X^{(qc)} + X^{(a)}, \quad X^{(a)} := I_{\Gamma_X} \cdot X, \quad X^{(qc)} := X - X^{(a)}. \quad (1.3)$$

The process $X^{(a)}$ (the accessible part of X) is a thin process with predictable jumps only, while $X^{(qc)}$ is a \mathbb{H} -quasi-left-continuous process (the quasi-left-continuous part of X).

Lemma 1.7. *Let X be an \mathbb{H} -semimartingale. Then X satisfies NUPBR(\mathbb{H}) if and only if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR(\mathbb{H}).*

Proof. Thanks to Proposition 1.3, X satisfies NUPBR(\mathbb{H}) if and only if there exist an \mathbb{H} -predictable real-valued process $\phi > 0$ and a positive \mathbb{H} -local martingale Y such that $Y(\phi \cdot X)$ is an \mathbb{H} -local martingale. Then, it is obvious that $Y(\phi I_{\Gamma_X} \cdot X)$ and $Y(\phi I_{\Gamma_X^c} \cdot X)$ are both \mathbb{H} -local martingales. This proves that $X^{(a)}$ and $X^{(qc)}$ both satisfy NUPNR(\mathbb{H}).

Conversely, if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPNR(\mathbb{H}), then there exist two \mathbb{H} -predictable real-valued processes $\phi_1, \phi_2 > 0$ and two positive \mathbb{H} -local martingales $D_1 = \mathcal{E}(N_1), D_2 = \mathcal{E}(N_2)$ such that $D_1(\phi_1 \cdot (I_{\Gamma_X} \cdot S))$ and $D_2(\phi_2 \cdot (I_{\Gamma_X^c} \cdot X))$ are both \mathbb{H} -local martingales. Remark that there is no loss of generality in assuming $N_1 = I_{\Gamma_X} \cdot N_1$ and $N_2 = I_{\Gamma_X^c} \cdot N_2$. Put

$$N := I_{\Gamma_X} \cdot N_1 + I_{\Gamma_X^c} \cdot N_2 \quad \text{and} \quad \psi := \phi_1 I_{\Gamma_X} + \phi_2 I_{\Gamma_X^c}.$$

Obviously, $\mathcal{E}(N) > 0$, $\mathcal{E}(N)$ and $\mathcal{E}(N)(\psi \cdot S)$ are \mathbb{H} -local martingales, ψ is \mathbb{H} -predictable and $0 < \psi \leq 1$. This ends the proof of the lemma. \square

Therefore, throughout the paper S is assumed to be a thin \mathbb{F} -semimartingale. This paper is organized as follows. The next section (Section 2) addresses the case of stopping at τ (i.e. deals with the model (S^τ, \mathbb{G})), while Section 3 focuses on the model $(S - S^\tau, \mathbb{G})$. Sections 4 and 5 prove the main results elaborated in Sections 2 and 3. Section 5 is the most technical part of the paper. We conclude this paper with an appendix, where we recall some useful technical results.

2 The Case of Stopping at τ

This section elaborates our results on the NUPBR for the model (S^τ, \mathbb{G}) in two subsections. The first subsection presents our principal results as well as their immediate consequences and/or applications, while the second subsection outlines a method to construct explicitly \mathbb{G} -local martingale deflators from \mathbb{F} -local martingale deflators. To this end, in addition to \mathbb{G} and D defined in (1.1), we associate to τ two important \mathbb{F} -supermartingales given by

$$Z_t := P(\tau > t \mid \mathcal{F}_t) \quad \text{and} \quad \tilde{Z}_t := P(\tau \geq t \mid \mathcal{F}_t). \quad (2.4)$$

The supermartingale Z is right-continuous with left limits and coincides with the \mathbb{F} -optional projection of $I_{\llbracket 0, \tau \rrbracket}$, while \tilde{Z} admits right limits and left limits only and is the \mathbb{F} -optional projection of $I_{\llbracket 0, \tau \rrbracket}$. The decomposition of Z leads to an important \mathbb{F} -martingale m , given by

$$m := Z + D^{o, \mathbb{F}}, \quad (2.5)$$

where $D^{o, \mathbb{F}}$ is the \mathbb{F} -dual optional projection of D (see [23] for more details).

2.1 The main results

In this subsection, we outline the main results on the NUPBR condition for the stopped thin \mathbb{F} -semimartingales (with predictable jumps only) with τ . To this end, we start by addressing the case of single jump processes with \mathbb{F} -predictable stopping times.

Theorem 2.1. *Consider an \mathbb{F} -predictable stopping time T and an \mathcal{F}_T -measurable variable ξ satisfying $E(|\xi||\mathcal{F}_{T-}) < +\infty$ P -a.s. on $\{T < +\infty\}$.*

If $S := \xi I_{\{Z_{T-} > 0\}} I_{[T, +\infty[}$, then the following assertions are equivalent:

- (a) S^τ satisfies $NUPBR(\mathbb{G})$.
- (b) The process $\tilde{S} := \xi I_{\{\tilde{Z}_T > 0\}} I_{[T, +\infty[} = I_{\{\tilde{Z} > 0\}} \cdot S$ satisfies $NUPBR(\mathbb{F})$.
- (c) S satisfies $NUPBR(\mathbb{F}, \tilde{Q}_T)$, where \tilde{Q}_T is

$$\tilde{Q}_T := \left(\frac{\tilde{Z}_T}{Z_{T-}} I_{\{Z_{T-} > 0\}} + I_{\{Z_{T-} = 0\}} \right) \cdot P, \quad (2.6)$$

- (d) S satisfies $NUPBR(\mathbb{F}, Q_T)$, where Q_T is defined by

$$\frac{dQ_T}{dP} := \frac{I_{\{\tilde{Z}_T > 0\} \cap \Gamma_0(T)}}{P(\tilde{Z}_T > 0 | \mathcal{F}_{T-})} + I_{\Omega \setminus \Gamma_0(T)}, \quad \Gamma_0(T) := \{P(\tilde{Z}_T > 0 | \mathcal{F}_{T-}) > 0\}. \quad (2.7)$$

The proof of this theorem is long and requires a result from the next subsection. Thus, this proof is delegated to Section 4.

Remark 2.2. (a) The importance of Theorem 2.1 goes beyond its vital role, as a building block for the more general result. In fact, Theorem 2.1 provides two different characterizations for the $NUPBR(\mathbb{G})$ of S^τ . The characterizations (c) and (d) are expressed in term of the $NUPBR(\mathbb{F})$ of S under absolute continuous change of measure, while the characterization (a) uses transformation of S without any change of measure. Furthermore, Theorem 2.1 can be easily extended to the case of countably many ordered predictable jump times $T_0 = 0 \leq T_1 \leq T_2 \leq \dots$ with $\sup_n T_n = +\infty$ P -a.s..

(b) In Theorem 2.1, the choice of S having the form $S := \xi I_{\{Z_{T-} > 0\}} I_{[T, +\infty[}$ is not restrictive. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := \xi I_{[T, +\infty[} = \xi I_{\{Z_{T-} > 0\}} I_{[T, +\infty[} + \xi I_{\{Z_{T-} = 0\}} I_{[T, +\infty[} =: \bar{S} + \hat{S}.$$

Thanks to $\{T \leq \tau\} \subset \{Z_{T-} > 0\}$, we have $\hat{S}^\tau = \xi I_{\{Z_{T-} = 0\}} I_{\{T \leq \tau\}} I_{[T, +\infty[} \equiv 0$ is (obviously) a \mathbb{G} -martingale. Thus, the only part of S that requires careful attention is $\bar{S} := \xi I_{\{Z_{T-} > 0\}} I_{[T, +\infty[}$.

The following proposition describes the models of τ for which any single jump \mathbb{F} -martingale (that jumps at fixed \mathbb{F} -predictable stopping time T), stopped at τ , satisfies the $NUPBR(\mathbb{G})$.

Proposition 2.3. *Let T be an \mathbb{F} -predictable stopping time. Then, the following assertions are equivalent:*

- (a) On $\{T < +\infty\}$, we have

$$\{\tilde{Z}_T = 0\} \subset \{Z_{T-} = 0\}. \quad (2.8)$$

- (b) For any $M := \xi I_{[T, +\infty[}$ where $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi | \mathcal{F}_{T-}) = 0$, M^τ satisfies $NUPBR(\mathbb{G})$.

Proof. We start by proving (a) \Rightarrow (b). Suppose that (2.8) holds. Then, due to Remark 2.2–(b), we can restrict our attention to the case where $M := \xi I_{\{Z_{T-} > 0\}} I_{[T, +\infty[}$ with $\xi \in L^\infty(\mathcal{F}_T)$ and $E(\xi | \mathcal{F}_{T-}) = 0$. Since assertion (a) is equivalent to $\llbracket T \rrbracket \cap \{\tilde{Z} = 0 \text{ \& } Z_- > 0\} = \emptyset$, we deduce that

$$\widetilde{M} := \xi I_{\{\tilde{Z}_T > 0\}} I_{\{Z_{T-} > 0\}} I_{[T, +\infty[} = M \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Therefore, a direct application of Theorem 2.1 (to M) allows us to conclude that M^τ satisfies the NUPBR(\mathbb{G}). This ends the proof of (a) \Rightarrow (b). To prove the reverse implication, we suppose that assertion (b) holds and consider

$$M := \xi I_{[T, +\infty[}, \quad \text{where } \xi := \left(I_{\{\tilde{Z}_T = 0\}} - P(\tilde{Z}_T = 0 | \mathcal{F}_{T-}) \right) I_{\{T < +\infty\}}.$$

Since $\{T \leq \tau\} \subset \{\tilde{Z}_T > 0\} \subset \{Z_{T-} > 0\}$, then we get

$$M^\tau = -P(\tilde{Z}_T = 0 | \mathcal{F}_{T-}) I_{\{T \leq \tau\}} I_{[T, +\infty[},$$

and this process is \mathbb{G} -predictable. Therefore, M^τ satisfies NUPBR(\mathbb{G}) if and only if it is a constant process equal to $M_0 = 0$ (see Lemma 1.6). This is equivalent to

$$0 = E \left[P(\tilde{Z}_T = 0 | \mathcal{F}_{T-}) I_{\{T \leq \tau\}} I_{[T, +\infty[} \right] = E \left(Z_{T-} I_{\{\tilde{Z}_T = 0 \text{ \& } T < +\infty\}} \right).$$

It is obvious that this equality is equivalent to (2.8), and assertion (a) follows. This ends the proof of the theorem. \square

The next theorem is an extension of Theorem 2.1 to the case where there are countable many arbitrary predictable jumps, and constitutes our first main result for the general thin semimartingales with predictable jumps only.

Theorem 2.4. *Let S be a thin process with predictable jump times only. Then, the following assertions are equivalent.*

(a) *The process S^τ satisfies the NUPBR(\mathbb{G}).*

(b) *For any $\delta > 0$, there exists a positive \mathbb{F} -local martingale, Y , such that ${}^{p, \mathbb{F}} \left(Y | \Delta S | I_{\{\tilde{Z} > 0\}} \right) < +\infty$ P -a.s. on $\{Z_- \geq \delta\}$ and*

$${}^{p, \mathbb{F}} \left(Y \Delta S I_{\{\tilde{Z} > 0\}} \right) I_{\{Z_- \geq \delta\}} = 0. \quad (2.9)$$

(c) *For any δ , the process*

$$S^{(0)} := \sum \Delta S I_{\{\tilde{Z} > 0 \text{ \& } Z_- \geq \delta\}} = I_{\{Z_- \geq \delta\}} \cdot S - \sum \Delta S I_{\{\tilde{Z} = 0 \text{ \& } Z_- \geq \delta\}}, \quad (2.10)$$

satisfies the NUPBR(\mathbb{F}).

The proof of this theorem is technically involved, especially the proof of (a) \Rightarrow (c), and thus it is postponed to Subsection 4.1.

Remark 2.5. *It is important to notice that, in Theorem 2.4, we did not assume any arbitrage condition on S . Therefore, as consequence, we obtain the following. Suppose that S is a thin process –with predictable jumps only– satisfying NUPBR(\mathbb{F}) and*

$$\{\tilde{Z} = 0 \text{ \& } Z_- > 0\} \cap \{\Delta S \neq 0\} = \emptyset.$$

Then, S^τ satisfies NUPBR(\mathbb{G}). This follows immediately from Theorem 2.4 by using $Y \in \mathcal{L}(S, \mathbb{F})$ and Lemma 1.4.

The following extends Proposition 2.3 to the case of countably many jumps that might not be ordered in any way.

Theorem 2.6. *The following assertions are equivalent.*

- (a) *The set $\{\tilde{Z} = 0 > Z_-\}$ is totally inaccessible.*
- (b) *X^τ satisfies the NUPBR(\mathbb{G}) for any thin process X with predictable jumps satisfying NUPBR(\mathbb{F}).*

Proof. The proof of the theorem will be achieved in two parts, namely part 1) and part 2) where we prove (b) \implies (a) and (a) \implies (b) respectively.

1) Suppose that assertion (b) holds. Then, thanks to Proposition 2.3, we deduce that for any \mathbb{F} -predictable stopping time T ,

$$\llbracket T \rrbracket \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset \quad (2.11)$$

on the one hand. On the other hand, since $\{\tilde{Z} = 0 < Z_-\}$ is thin, there exists a sequence of \mathbb{F} -stopping times $(\sigma_k)_{k \geq 1}$ with disjoint graphs such that

$$\{\tilde{Z} = 0 < Z_-\} = \bigcup_{k=1}^{+\infty} \llbracket \sigma_k \rrbracket. \quad (2.12)$$

Recall that, for each σ_k , there exist two \mathbb{F} -stopping times $(\sigma_k^i$ and σ_k^a that are totally inaccessible and accessible respectively) and a sequence of \mathbb{F} -predictable stopping times $(T_l^{(k)})_{l \geq 1}$ such that

$$\llbracket \sigma_k \rrbracket = \llbracket \sigma_k^i \rrbracket \cup \llbracket \sigma_k^a \rrbracket, \quad \llbracket \sigma_k^a \rrbracket \subset \bigcup_{l=1}^{+\infty} \llbracket T_l^{(k)} \rrbracket.$$

Thus, by combining these with $\left(\bigcup_{k=1}^{+\infty} \llbracket \sigma_k^i \rrbracket \right) \cap \left(\bigcup_{k=1, l=1}^{+\infty} \llbracket T_l^{(k)} \rrbracket \right) = \emptyset$, (2.12) and (2.11), we derive

$$\bigcup_{k=1}^{+\infty} \llbracket \sigma_k^a \rrbracket = \left(\bigcup_{k=1, l=1}^{+\infty} \llbracket T_l^{(k)} \rrbracket \right) \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset.$$

This proves that $\{\tilde{Z} = 0 < Z_-\}$ is a totally inaccessible set and the proof of (b) \implies (a) is completed.

2) To prove the reverse sense, we assume that assertion (a) holds, and consider $X = \sum \xi_n I_{[T_n, +\infty[}$ satisfying NUPBR(\mathbb{F}), where T_n is an \mathbb{F} -predictable stopping time and ξ_n is a bounded \mathcal{F}_{T_n} -measurable

random variable. Since $\{\Delta X \neq 0\} = \bigcup_{n=1}^{+\infty} \llbracket T_n \rrbracket$ is predictable, we get $\{\tilde{Z} = 0 < Z_-\} \cap \{\Delta X \neq 0\} = \emptyset$,

and hence, from Remark 2.5, X^τ satisfies the NUPBR(\mathbb{G}). This ends the proof of the theorem. \square

The complete general result, in this spirit of describing the model for τ that preserves the NUPBR after stopping with τ , is the following.

Theorem 2.7. *The following assertions are equivalent.*

- (a) *The set $\{\tilde{Z} = 0 > Z_-\}$ is evanescent.*
- (b) *X^τ satisfies the NUPBR(\mathbb{G}) for any X satisfying the NUPBR(\mathbb{F}).*

Proof. The proof follows immediately from the combination of Theorem 2.6 and Proposition 2.22 in [3] (where the authors prove that the thin set $\{\tilde{Z} = 0 < Z_-\}$ is accessible if and only if assertion (b) above holds for any \mathbb{F} -quasi-left-continuous process X). \square

2.2 Explicit local martingale deflators

This section discusses how to construct explicitly \mathbb{G} -local martingale deflators from \mathbb{F} -deflators for a class of processes. This is achieved, for single jump processes and general thin processes afterwards, by considering \mathbb{F} -neutralized processes.

Proposition 2.8. *Let $M := \xi I_{\llbracket T, +\infty \rrbracket}$ be an \mathbb{F} -martingale, where T is an \mathbb{F} -predictable stopping time, and ξ is an \mathcal{F}_T -measurable random variable. Then the following assertions are equivalent.*

- (a) *M is an \mathbb{F} -martingale under Q_T given by (2.7).*
- (b) *On the set $\{T < +\infty\}$, we have*

$$E \left(M_T I_{\{\tilde{Z}_T=0\}} | \mathcal{F}_{T-} \right) = 0, \quad P - a.s. \quad (2.13)$$

- (c) *M^τ is a \mathbb{G} -martingale under $Q_T^\mathbb{G}$ given by*

$$\frac{dQ_T^\mathbb{G}}{dP} := \frac{U^\mathbb{G}(T)}{E(U^\mathbb{G}(T) | \mathcal{G}_{T-})} \text{ where } U^\mathbb{G}(T) := I_{\{T > \tau\}} + I_{\{T \leq \tau\}} \frac{Z_{T-}}{\tilde{Z}_T}. \quad (2.14)$$

Proof. The proof will be achieved in two steps where we prove (a) \iff (b) and (a) \iff (c) respectively.

Step 1. Here, we prove (a) \iff (b). For simplicity we denote by $Q := Q_T$, where Q_T is defined in (2.7), and remark that on $\{Z_{T-} = 0\}$, Q coincides with P and (2.13) holds, due to $\{Z_{T-} = 0\} \subset \{\tilde{Z}_T = 0\}$. Thus, it is enough to prove (a) \iff (2.13) on the set $\{T < +\infty \text{ \& } Z_{T-} > 0\}$. On this set, due to $E(\xi | \mathcal{F}_{T-}) = 0$ (since M is an \mathbb{F} -martingale), we derive

$$\begin{aligned} E^Q(\xi | \mathcal{F}_{T-}) &= E(\xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}) \left(P(\tilde{Z}_T > 0 | \mathcal{F}_{T-}) \right)^{-1} \\ &= -E(\xi I_{\{\tilde{Z}_T = 0\}} | \mathcal{F}_{T-}) \left(P(\tilde{Z}_T > 0 | \mathcal{F}_{T-}) \right)^{-1}. \end{aligned}$$

Therefore, assertion (a) (or equivalently $E^Q(\xi | \mathcal{F}_{T-}) = 0$) is equivalent to (2.13). This ends the proof of (a) \iff (b).

Step 2. To prove (a) \iff (c), we notice that due to $\{T \leq \tau\} \subset \{\tilde{Z}_T > 0\} \subset \{Z_{T-} > 0\}$, on $\{T \leq \tau\}$ we have

$$\begin{aligned} P \left(\tilde{Z}_T > 0 | \mathcal{F}_{T-} \right) E^{Q_T^\mathbb{G}}(\xi | \mathcal{G}_{T-}) &= E \left(\frac{Z_{T-}}{\tilde{Z}_T} \xi I_{\{T \leq \tau\}} | \mathcal{G}_{T-} \right) = E \left(\xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-} \right) \\ &= E^Q(\xi | \mathcal{F}_{T-}) P \left(\tilde{Z}_T > 0 | \mathcal{F}_{T-} \right). \end{aligned}$$

This equality proves that $M^\tau \in \mathcal{M}(Q^\mathbb{G}, \mathbb{G})$ if and only if $M \in \mathcal{M}(Q, \mathbb{F})$, and the proof of (a) \iff (c) is completed. This ends the proof of the theorem. \square

To generalize this proposition to the case of infinitely many jumps that might not be ordered at all, we need to introduce some notations and recall some facts from [3]. First of all, we refer to [12] (Chapter VIII.2 sections 32-35 pages 356-361) and [21] (Chapter III.4.b, Definition 3(3.8), pages 106-109) for the optional stochastic integration (see also Definition 3.4 in [3]).

Definitions 2.9. *Let N be an \mathbb{H} -local martingale with continuous part N^c and K be an \mathbb{H} -optional process. K is said to be integrable with respect to N if ${}^{p, \mathbb{H}}(K)$ is N^c -integrable, ${}^{p, \mathbb{H}}(K | \Delta N) < +\infty$ and*

$$\left(\sum (K \Delta N - {}^{p, \mathbb{H}}(K \Delta N)) \right)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{H}).$$

Put

$$K^{\mathbb{G}} := \frac{Z_-^2 \tilde{Z}^{-1}}{Z_-^2 + \Delta \langle m \rangle^{\mathbb{F}}} I_{[0, \tau]}, \quad V^{\mathbb{G}} := \sum p, \mathbb{F} (I_{\{\tilde{Z}=0\}} I_{[0, \tau]}), \quad (2.15)$$

and to any \mathbb{F} -local martingale M , we associate the \mathbb{G} -local martingale part of M^τ given by

$$\widehat{M} := M^\tau - Z_-^{-1} I_{[0, \tau]} \cdot \langle M, m \rangle^{\mathbb{F}}. \quad (2.16)$$

Below, we recall some useful results of [3].

Proposition 2.10. *The following assertions hold.*

(a) *The \mathbb{G} -optional process $K^{\mathbb{G}}$ is \widehat{m} -integrable in the sense of the above definition. Here $\widehat{m} := m^\tau - Z_-^{-1} I_{[0, \tau]} \cdot \langle m \rangle^{\mathbb{F}}$. Furthermore the resulting integral*

$$\tilde{L}^{(b)} := \mathcal{E} \left(-\frac{K^{\mathbb{G}}}{1 - \Delta V^{\mathbb{G}}} \odot \widehat{m} \right), \quad (2.17)$$

is a positive (i.e. $\tilde{L}^{(b)} > 0$) \mathbb{G} -local martingale satisfying $[\tilde{L}^{(b)}, M] \in \mathcal{A}_{loc}(\mathbb{G})$ for any \mathbb{F} -local martingale M .

(b) *$V^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G})$ and $(1 - \Delta V^{\mathbb{G}})^{-1}$ is \mathbb{G} -locally bounded.*

The proof of this proposition can be found in [3] (see Lemma 3.3 and Proposition 3.6). The extension of Proposition 2.8 goes through connecting the random variable $U^{\mathbb{G}}(T)$ defined in (2.14) to the process $\tilde{L}^{(a)}$ as follows.

Remark 2.11. *In virtue of the calculation performed in [3] (see equation (B.1) where the authors calculate the jumps of $K^{\mathbb{G}} \odot \widehat{m}$), we have*

$$-\frac{(1 - \Delta V^{\mathbb{G}}) \Delta \tilde{L}^{(b)}}{\tilde{L}^{(b)}} = K^{\mathbb{G}} \Delta \widehat{m} - p, \mathbb{G} \left(K^{\mathbb{G}} \Delta \widehat{m} \right) = \frac{\Delta m}{\tilde{Z}} I_{[0, \tau]} - \Delta V^{\mathbb{G}}.$$

Thus, for an \mathbb{F} -predictable stopping time T , on $\{T \leq \tau\}$ we get

$$U_T^{\mathbb{G}} = \frac{Z_{T-}}{\tilde{Z}_T} = (1 - \Delta V_T^{\mathbb{G}}) \frac{\tilde{L}_T^{(b)}}{\tilde{L}_{T-}^{(b)}}.$$

This proves that assertions (a) and (b) of Proposition 2.8 are equivalent to

$$\tilde{L}^{(b)} M^\tau \text{ is a } \mathbb{G}\text{-martingale for any single jump } \mathbb{F}\text{-martingale } M. \quad (2.18)$$

Theorem 2.12. *Consider $\tilde{L}^{(b)}$ defined in (2.17) and let M be a thin \mathbb{F} -martingale satisfying*

$$p, \mathbb{F} \left(\Delta M I_{\{\tilde{Z}=0 < Z_-\}} \right) \equiv 0. \quad (2.19)$$

Then, $\tilde{L}^{(b)} M^\tau$ is a \mathbb{G} -local martingale.

Proof. We start by remarking that it is enough to prove that there exists a \mathbb{G} -predictable process φ such that $0 < \varphi \leq 1$ and $\tilde{L}^{(b)}(\varphi \cdot M^\tau)$ is a \mathbb{G} -martingale (local martingale). This means that $\tilde{L}^{(b)} \in \mathcal{L}(M^\tau, \mathbb{G})$ (i.e it is a σ -martingale density for M^τ under \mathbb{G}). This remark that simplifies the proof based on the fact that $[\tilde{L}^{(b)}, M^\tau]$ is locally integrable and Proposition 3.3 and Corollary 3.5 of

[7]. Again, thanks to $[\tilde{L}^{(b)}, M^\tau] \in \mathcal{A}_{loc}(\mathbb{G})$, we deduce that ${}^{p,\mathbb{G}}\left(\tilde{L}^{(b)}|\Delta M^\tau|\right) < +\infty$, and consider the following \mathbb{G} -predictable process

$$\phi := \left[1 + {}^{p,\mathbb{G}}(|\Delta M^\tau|) + {}^{p,\mathbb{G}}\left(\tilde{L}^{(b)}|\Delta M^\tau|\right)\right]^{-1} \left[I_{\Omega \setminus (\cup_n \llbracket T_n \rrbracket)} + \sum_{n=1}^{+\infty} 2^{-n} I_{\llbracket T_n \rrbracket}\right],$$

where $(T_n)_{n \geq 1}$ is the sequence of \mathbb{F} -predictable stopping times that exhausts the jumps of M . Thus, it is easy to check that $0 < \phi \leq 1$, and both processes $\phi \cdot M^\tau$ and $\tilde{L}_-^{(b)}\phi \cdot M^\tau + [\tilde{L}^{(b)}, \phi \cdot M^\tau] = \sum \tilde{L}^{(b)}\phi \Delta M^\tau$ have integrable variations on the one hand. On the other hand, since $\sum \tilde{L}^{(b)}\phi \Delta M^\tau$ jumps on predictable stopping times only, its \mathbb{G} -compensator is

$$\sum {}^{p,\mathbb{G}}\left(\tilde{L}^{(b)}\phi \Delta M^\tau\right) = \sum \phi {}^{p,\mathbb{G}}\left(\tilde{L}^{(b)}\Delta M^\tau\right) \equiv 0.$$

This proves that $\tilde{L}_-^{(b)}\phi \cdot M^\tau + [\tilde{L}^{(b)}, \phi \cdot M^\tau]$ is a \mathbb{G} -local martingale or equivalently $\tilde{L}^{(b)}(\phi \cdot M^\tau)$ is a \mathbb{G} -local martingale. This ends the proof of the theorem. \square

Corollary 2.13. *For any thin \mathbb{F} -martingale M such that $\{\Delta M \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$ is evanescent, $\tilde{L}^{(b)}M^\tau$ is a \mathbb{G} -local martingale.*

Proof. The proof of the corollary follows immediately from Theorem 2.12, as the condition $\{\Delta M \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset$ implies (2.19). \square

3 The part after τ

Herein, we focus on the process $S - S^\tau$, and in the same spirit of Section 2 we summarize results in two subsections. The first subsection outlines the principal results, while the second subsection explains how to obtain \mathbb{G} -local martingale deflators for $S - S^\tau$ from the \mathbb{F} -deflators of S when S varies in a class of processes. However in this section we consider the following assumption on τ

$$\tau \text{ is an honest time and } Z_\tau < 1 \quad P - a.s. \quad (3.20)$$

3.1 The main results

This subsection presents our main results on the NUPBR for $(S - S^\tau, \mathbb{G})$. These results are elaborated for single jump processes and general thin processes with predictable jumps only as well.

Theorem 3.1. *Suppose that τ is an honest time. Consider an \mathbb{F} -predictable stopping time T and an \mathcal{F}_T -measurable r.v. ξ such that $E(|\xi| | \mathcal{F}_{T-}) < +\infty$ P -a.s. on $\{T < +\infty\}$.*

If $S := \xi I_{\{Z_{T-} < 1\}} I_{\llbracket T, +\infty \rrbracket}$, then the following are equivalent:

- (a) $S - S^\tau$ satisfies the NUPBR(\mathbb{G}).
- (b) S satisfies the NUPBR(\mathbb{F}, \tilde{Q}'_T), where

$$\tilde{Q}'_T := \left(\frac{1 - \tilde{Z}_T}{1 - Z_{T-}} I_{\{Z_{T-} < 1\}} + I_{\{Z_{T-} = 1\}} \right) \cdot P. \quad (3.21)$$

- (c) S satisfies the NUPBR(\mathbb{F}, Q'_T), where for $\Gamma_1(T) := \{P(\tilde{Z}_T < 1 | \mathcal{F}_{T-}) > 0 \text{ \& } T < +\infty\}$ we set

$$Q'_T := \left(\frac{I_{\{\tilde{Z}_T < 1\} \cap \Gamma_1(T)}}{P(\tilde{Z}_T < 1 | \mathcal{F}_{T-})} + I_{\Omega \setminus \Gamma_1(T)} \right) \cdot P. \quad (3.22)$$

- (d) $\tilde{S} := \xi I_{\{\tilde{Z}_T < 1\}} I_{\llbracket T, +\infty \rrbracket}$ satisfies the NUPBR(\mathbb{F}).

The proof of this theorem is long and requires intermediary results. Thus, we postpone the proof to Subsection 4.1.

Remark 3.2. *Theorem 3.1 provides two equivalent (and conceptually different) characterisations for the condition that $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$. One of these characterisations uses the $\text{NUPBR}(\mathbb{F})$ property under P for a transformation of S , while the other characterisation is essentially based on the $\text{NUPBR}(\mathbb{F})$ for S under an absolutely continuous probability measure.*

The next theorem describes the models for τ that preserve the $\text{NUPBR}(\mathbb{G})$ after τ for any single jump \mathbb{F} -martingale.

Theorem 3.3. *Suppose that τ is an honest and consider an \mathbb{F} -predictable stopping time T . Then, the following assertions are equivalent:*

(a) *On $\{T < +\infty\}$, we have*

$$\{\tilde{Z}_T = 1\} \subset \{Z_{T-} = 1\}. \quad (3.23)$$

(b) *For any $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi | \mathcal{F}_{T-}) = 0$ P -a.s. on $\{T < +\infty\}$, the process $M - M^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$, where $M := \xi I_{[T, +\infty[}$.*

Proof. Suppose that assertion (a) holds, and consider $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi | \mathcal{F}_{T-}) = 0$, P -a.s. on $\{T < +\infty\}$. By decomposing M into

$$M = I_{\{Z_{T-} < 1\}} \xi I_{[T, +\infty[} + I_{\{Z_{T-} = 1\}} \xi I_{[T, +\infty[} := M^{(1)} + M^{(2)},$$

and noting that $M^{(2)} - (M^{(2)})^\tau = 0$, we can restrict our attention to the case where $M = M^{(1)}$ on the one hand. On the other hand, since $\{Z_{T-} = 1\} \subset \{\tilde{Z}_T = 1\}$ P -a.s. on $\{T < +\infty\}$, it is obvious that (3.23) implies $\{\tilde{Z}_T < 1\} = \{Z_{T-} < 1\}$ on $\{T < +\infty\}$, and hence

$$\tilde{M} := I_{\{\tilde{Z}_T < 1\}} M = M \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Thus, assertion (b) follows from a direct application of Theorem 3.1 to M . This ends the proof of (a) \Rightarrow (b). To prove the converse, we assume that assertion (b) holds, and we consider the \mathcal{F}_T -measurable and bounded r.v. $\xi := (I_{\{\tilde{Z}_T = 1\}} - P(\tilde{Z}_T = 1 | \mathcal{F}_{T-})) I_{\{T < +\infty\}}$ and the bounded \mathbb{F} -martingale $M := \xi I_{[T, +\infty[}$. Then, on the one hand, $M - M^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$. On the other hand, due to $\{T > \tau\} \subset \{\tilde{Z}_T < 1\}$, the finite variation process

$$M - M^\tau = -P(\tilde{Z}_T = 1 | \mathcal{F}_{T-}) I_{\{T > \tau\}} I_{[T, +\infty[} \quad \text{is } \mathbb{G}\text{-predictable.}$$

Thus, it is null, or equivalently $\{Z_{T-} < 1\} \subset \{\tilde{Z}_T < 1\}$ P -a.s. on $\{T < +\infty\}$. This proves assertion (a), and the proof of the theorem is completed. \square

The following extends Theorem 3.1 to the case of general thin processes.

Theorem 3.4. *Suppose that τ satisfies (3.20), and S is a thin process with predictable jumps only. Then, the following assertions are equivalent.*

(a) *The process $S - S^\tau$ satisfies the $\text{NUPBR}(\mathbb{G})$.*

(b) *For any $\delta > 0$, there exists a positive \mathbb{F} -local martingale Y , such that*

$${}^{p, \mathbb{F}} \left(Y |\Delta S| I_{\{\tilde{Z} < 1\}} \right) < +\infty \quad \& \quad {}^{p, \mathbb{F}} \left(Y \Delta S I_{\{\tilde{Z} < 1\}} \right) = 0 \quad \text{on } \{1 - Z_- \geq \delta\}. \quad (3.24)$$

(c) *For any δ , the process*

$$S^{(1)} := \sum \Delta S I_{\{\tilde{Z} < 1 \text{ \& } 1 - Z_- \geq \delta\}}, \quad (3.25)$$

satisfies the $\text{NUPBR}(\mathbb{F})$.

The proof of this theorem is long and is based on a result of the next subsection. Thus, this proof is postponed to Subsection 5.2.

Remark 3.5. 1) The process $S^{(1)}$ defined in (3.25) is a thin semimartingale. In fact, we have $S^{(1)} = I_{\{1-Z_- \geq \delta\}} \cdot S - \sum \Delta S I_{\{\tilde{Z}=1 \text{ \& } 1-Z_- \geq \delta\}}$, and

$$\sum I_{\{\tilde{Z}=1 \text{ \& } 1-Z_- \geq \delta\}} \leq \delta^{-2} \sum (\Delta m)^2 \leq \delta^{-2} [m, m] \in \mathcal{A}_{loc}^+(\mathbb{F}).$$

2) The proof of (a) \implies (b) is the very technical part in the proof of the theorem, while the rest is easy and is postponed to keep this section short.

Theorem 3.6. The following assertions are equivalent.

- (a) The set $\{\tilde{Z} = 1 > Z_-\}$ is totally inaccessible.
- (b) $X - X^\tau$ satisfies the NUPBR(\mathbb{G}) for any thin process X with predictable jumps satisfying NUPBR(\mathbb{F}).

Proof. Suppose that assertion (a) holds, and consider a thin process with predictable jumps, X , satisfying NUPBR(\mathbb{F}). Thus, $\{\Delta X \neq 0\}$ is a thin accessible set, and hence $\{\tilde{Z} = 1 > Z_-\} \cap \{\Delta X \neq 0\} = \emptyset$. Therefore, we conclude that

$$X^{(1)} := \sum \Delta X I_{\{\tilde{Z} < 1 \text{ \& } 1-Z_- \geq \delta\}} = I_{\{1-Z_- \geq \delta\}} \cdot X \text{ satisfies NUPBR}(\mathbb{F}).$$

Then, a direct application of Theorem 3.4 leads to the NUPBR(\mathbb{G}) of $X - X^\tau$. This proves (a) \implies (b). To prove the reverse, we remark that the set $\{\tilde{Z} = 1 > Z_-\}$ is thin, and we mimic exactly the part 1) of the proof of Theorem 2.6. This ends the proof of theorem. \square

Theorem 3.7. The following assertions are equivalent.

- (a) The set $\{\tilde{Z} = 1 > Z_-\}$ is evanescent.
- (b) $X - X^\tau$ satisfies the NUPBR(\mathbb{G}) for any X satisfying NUPBR(\mathbb{F}).

Proof. The proof follows immediately from the combination of Theorem 3.6 and Proposition 2.18 in [4] (where the authors prove that the this set $\{\tilde{Z} = 1 > Z_-\}$ is accessible if and only if assertion (b) of the theorem above holds for any quasi-left-continuous process X (i.e. X does not jump on predictable stopping times)). \square

3.2 Explicit construction of local martingale deflators

To construct \mathbb{G} -deflators for thin \mathbb{F} -local martingale, we start by illustrating this construction for single jump \mathbb{F} -martingales.

Theorem 3.8. Let τ be an honest time. Consider an \mathbb{F} -predictable stopping time T and an \mathcal{F}_T -measurable r.v. ξ such that $E[|\xi| | \mathcal{F}_{T-}] < +\infty$, P -a.s. Define $M := \xi I_{\{Z_{T-} < 1\}} I_{[T, +\infty[}$,

$$\begin{aligned} \frac{dQ_T^\mathbb{F}}{dP} &:= D^\mathbb{F} := \frac{I_{\{\tilde{Z}_T < 1 \text{ \& } P(\tilde{Z}_T < 1 | \mathcal{F}_{T-}) > 0\}}}{P(\tilde{Z}_T < 1 | \mathcal{F}_{T-})} + I_{\{P(\tilde{Z}_T < 1 | \mathcal{F}_{T-}) = 0\}}, \quad \text{and} \\ \frac{dQ_T^\mathbb{G}}{dP} &:= D^\mathbb{G} := \frac{1 - Z_{T-}}{(1 - \tilde{Z}_T)P(\tilde{Z}_T < 1 | \mathcal{F}_{T-})} I_{\{T > \tau\}} + I_{\{T \leq \tau\}}. \end{aligned} \quad (3.26)$$

Then the following assertions are equivalent.

- (a) M is a $(Q_T^\mathbb{F}, \mathbb{F})$ -martingale.
- (b) On $\{Z_{T-} < 1\}$, we have

$$E\left(\xi I_{\{\tilde{Z}_T < 1\}} \mid \mathcal{F}_{T-}\right) = 0, \quad P - a.s. \quad (3.27)$$

- (c) $(M - M^\tau)$ is a $(Q_T^\mathbb{G}, \mathbb{G})$ -martingale.

Proof. For the sake of simplicity, throughout the proof, we put $Q_1 := Q_T^{\mathbb{F}}$ and $Q_2 := Q_T^{\mathbb{G}}$. The proof of the theorem will be given in two steps.

1) Here, we prove (a) \iff (b). Thanks to $\{\tilde{Z}_T < 1\} \subset \{Z_{T-} < 1\}$ and $E[D^{\mathbb{F}}|\mathcal{F}_{T-}] = 1$ on $\{T < +\infty\}$, we derive

$$E^{Q_1}[\xi I_{\{Z_{T-} < 1\}}|\mathcal{F}_{T-}] = E\left[D^{\mathbb{F}}\xi I_{\{Z_{T-} < 1\}}|\mathcal{F}_{T-}\right] = \frac{E\left[\xi I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}\right]}{P(\tilde{Z}_T < 1|\mathcal{F}_{T-})}I_{\{Z_{T-} < 1\}}.$$

Therefore, (a) \iff (b) follows from combining this equality and the fact that M is a (Q_1, \mathbb{F}) -martingale if and only if $E^{Q_1}(M_T | \mathcal{F}_{T-})I_{\{T < +\infty\}} = 0$.

2) Here, we prove (b) \iff (c). To this end, we first notice that $M - M^\tau = \xi I_{\{Z_{T-} < 1 \text{ \& } T > \tau\}}I_{\llbracket \tau, +\infty \rrbracket}$ is a (Q_2, \mathbb{G}) -martingale if and only if $E^{Q_2}[\xi I_{\{Z_{T-} < 1 \text{ \& } T > \tau\}}|\mathcal{G}_{T-}]I_{\{T < +\infty\}} = 0$. Then, using the fact that $E[D^{\mathbb{G}}|\mathcal{G}_{T-}] = 1$ on $\{T < +\infty\}$, we get

$$\begin{aligned} E^{Q_2}[\xi I_{\{Z_{T-} < 1 \text{ \& } T > \tau\}}|\mathcal{G}_{T-}] &= E\left[D^{\mathbb{G}}\xi I_{\{Z_{T-} < 1\}}I_{\{T > \tau\}}|\mathcal{G}_{T-}\right] \\ &= E\left[\frac{\xi I_{\{T > \tau\}}}{1 - \tilde{Z}_T}|\mathcal{G}_{T-}\right] \frac{1 - Z_{T-}}{P(\tilde{Z}_T < 1|\mathcal{F}_{T-})}I_{\{Z_{T-} < 1 \text{ \& } T > \tau\}} \\ &= \frac{E\left[\xi I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}\right]}{P(\tilde{Z}_T < 1|\mathcal{F}_{T-})}I_{\{Z_{T-} < 1\}}I_{\{T > \tau\}}, \end{aligned} \tag{3.28}$$

where the last equality in (3.28) follows from the fact that, τ being honest and

$$E(H | \mathcal{G}_{T-})I_{\{T > \tau\}} = E\left(H(1 - \tilde{Z}_T) | \mathcal{F}_{T-}\right)(1 - Z_{T-})^{-1}I_{\{T > \tau\}}.$$

for any \mathcal{F}_T -measurable random variable H such that the above conditional expectations exist (see Proposition 5.3 of [23]). Therefore, if assertion (b) holds, then assertion (c) follows immediately from (3.28). Conversely, if assertion (c) holds, then $E^{Q_2}[\xi I_{\{Z_{T-} < 1\}}I_{\{T > \tau\}}|\mathcal{G}_{T-}] = 0$. Thus, a combination of this with (3.28) leads to $E\left[\xi I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}\right](1 - Z_{T-}) = 0$. This proves assertion (b), and the proof of the theorem is completed. \square

Remark 3.9. Theorem 3.8 can be viewed as continuous-time version of Theorem 4.5 in [8], and it can be generalized easily to the case of a finite number of ordered \mathbb{F} -predictable stopping times on the one hand. On the other hand, when extending this theorem to the case of general thin semimartingales, the main difficulty lies in the fact of finding a positive \mathbb{F} -local martingale, L such that the density of $Q_T^{\mathbb{F}}$ defined in (3.26) coincides with L_T for any \mathbb{F} -predictable stopping time T . This difficulty remains an open problem and we are unable to see how to approach it. In contrast to $Q_T^{\mathbb{F}}$, the probability $Q_T^{\mathbb{G}}$ –given also in (3.26)– satisfies $dQ_T^{\mathbb{G}}/dP = \tilde{L}_T^{(a)}/\tilde{L}_{T-}^{(a)}$, where $\tilde{L}^{(a)}$ is a positive \mathbb{G} -local martingale that will be described below. To this end we need to introduce some notations and recall some results from [4].

Throughout the rest of this subsection, we consider the following notations for any $M \in \mathcal{M}_{loc}(\mathbb{F})$

$$\widehat{M}^{(a)} := M - M^\tau + (1 - Z_-)^{-1}I_{\llbracket \tau, +\infty \rrbracket} \cdot \langle m \rangle^{\mathbb{F}} \in \mathcal{M}_{loc}(\mathbb{G}), \tag{3.29}$$

$$W^{\mathbb{G}} := \sum p, \mathbb{F} \left(I_{\{\tilde{Z}=1\}} \right) I_{\llbracket \tau, +\infty \rrbracket}, \tag{3.30}$$

$$K^{(a)} := \frac{(1 - Z_-)^2(1 - \tilde{Z})^{-1}}{(1 - Z_-)^2 + \Delta \langle m \rangle^{\mathbb{F}}} I_{\llbracket \tau, +\infty \rrbracket}. \tag{3.31}$$

In the following, we recall a useful result from [4].

Proposition 3.10. *The following assertions hold.*

- (a) *The positive process $(1 - \Delta W^{\mathbb{G}})^{-1}$ is \mathbb{G} -locally bounded.*
- (b) *The \mathbb{G} -optional process, $K^{(a)}$, is $\widehat{m}^{(a)}$ -integrable (with respect to Definition 2.9). The resulting integral*

$$\widetilde{L}^{(a)} := \mathcal{E} \left(K^{(a)} (1 - \Delta W^{\mathbb{G}})^{-1} \odot \widehat{m}^{(a)} \right), \quad (3.32)$$

is a positive G -local martingale satisfying $[\widetilde{L}^{(a)}, \widehat{M}^{(a)}] \in \mathcal{A}_{loc}(\mathbb{G})$.

In order to extend Theorem 3.8 to the case of general thin semimartingales, we start by connecting the probability $Q_T^{\mathbb{G}}$ and $\widetilde{L}^{(a)}$ as follows.

Remark 3.11. *Put $L^{\mathbb{G}} := K^{(a)} \odot \widehat{m}^{(a)}$. Then, we derive*

$$\begin{aligned} D^{\mathbb{G}}(T) : &= \frac{1 - Z_{T-}}{1 - \widetilde{Z}_T} \frac{I_{\{T > \tau\}}}{P(\widetilde{Z}_T < 1 | \mathcal{F}_{T-})} + I_{\{T \leq \tau\}} = \left(1 + \frac{\Delta m_T}{1 - \widetilde{Z}_T} \right) I_{\{T > \tau\}} + I_{\{T \leq \tau\}} \\ &= \frac{1 + \Delta L^{\mathbb{G}} - \Delta V^{\mathbb{G}}}{1 - \Delta V^{\mathbb{G}}} = 1 + \Delta \widetilde{L}^{(a)} = \frac{\widetilde{L}_T^{(a)}}{\widetilde{L}_{T-}^{(a)}}. \end{aligned}$$

As a result, assertions (a) and (b) of Theorem 3.8 are equivalent to

$$\widetilde{L}^{(a)}(M - M^{\tau}) \text{ is a } \mathbb{G}\text{-martingale}, \quad (3.33)$$

for any single jump \mathbb{F} -martingale, M , with predictable jump time.

Now, we are at the stage of extending Theorem 3.8 to the general case of thin processes.

Theorem 3.12. *Let M be a thin \mathbb{F} -local martingale such that*

$${}^{p, \mathbb{F}} \left(\Delta M I_{\{\widetilde{Z}=1 > Z_{-}\}} \right) \equiv 0. \quad (3.34)$$

Then, $\widetilde{L}^{(a)}(M - M^{\tau})$ is a \mathbb{G} -local martingale.

Proof. Thanks to Itô's formula, it is immediate that $\widetilde{L}^{(a)}(M - M^{\tau})$ is a \mathbb{G} -local martingale if and only if

$$X^{\mathbb{G}} := M - M^{\tau} + [\widetilde{L}^{(a)}, M - M^{\tau}] \quad (3.35)$$

is a \mathbb{G} -local martingale. Since $X^{\mathbb{G}}$ is a \mathbb{G} -special semimartingale, hence it is enough to prove that $X^{\mathbb{G}}$ is a σ -martingale under \mathbb{G} . To prove this latter fact, thanks to Proposition 3.3 and Corollary 3.5 of [7], it is enough to prove that $\Phi \cdot X^{\mathbb{G}}$ is \mathbb{G} -local martingale for some \mathbb{G} -predictable process Φ such that $0 < \Phi \leq 1$. Since M is a thin process with predictable jump times only that we denote by $(T_n)_{n \geq 1}$, we get

$$X^{\mathbb{G}} = \sum \widetilde{L}^{(a)} \Delta M I_{\llbracket \tau, +\infty \rrbracket},$$

and jumps on the sequence of stopping times $(T_n)_{n \geq 1}$ only on the one hand. On the other hand, due to Proposition 3.10 (assertion (b)), we have ${}^{p, \mathbb{G}}(\widetilde{L}^{(a)} |\Delta M|) I_{\llbracket \tau, +\infty \rrbracket} < +\infty$, and hence the \mathbb{G} -predictable process

$$\Phi := \left[\sum I_{\llbracket T_n \rrbracket} 2^{-n} + I_{\Omega \setminus (\cup_n \llbracket T_n \rrbracket)} \right] \left(1 + {}^{p, \mathbb{G}}(\widetilde{L}^{(a)} |\Delta M|) I_{\llbracket \tau, +\infty \rrbracket} \right)^{-1},$$

satisfies $0 < \Phi \leq 1$, $\Phi \cdot X^{\mathbb{G}} \in \mathcal{A}(\mathbb{G})$, and its \mathbb{G} -compensator is given by

$$(X^{\mathbb{G}})^{p, \mathbb{G}} = \sum_n \Phi {}^{p, \mathbb{G}}(\widetilde{L}^{(a)} \Delta M^{(n)}) I_{\llbracket \tau, +\infty \rrbracket} = 0.$$

Here $M^{(n)} := \Delta M_{T_n} I_{\llbracket T_n, +\infty \rrbracket}$, while the last equality follows from (3.33) of Remark 3.11. This proves that $\Phi \cdot X^{\mathbb{G}}$ is a \mathbb{G} -local martingale, and the proof of the theorem is completed. \square

Corollary 3.13. *a) If M be a thin \mathbb{F} -local martingale such that $\{\Delta M \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$, then $\tilde{L}^{(a)}(M - M^\tau)$ is a \mathbb{G} -local martingale.
b) Suppose that S is thin, $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$, and S satisfies the NUPBR(\mathbb{F}). Then $S - S^\tau$ satisfies the NUPBR(\mathbb{G}).*

Proof. Since S satisfies the NUPBR(\mathbb{F}), then there exist an \mathbb{F} -predictable process ϕ , a sequence of \mathbb{F} -stopping times $(T_n)_{n \geq 1}$ that increases to infinity, and a probability measure $Q_n \sim P$ on $(\Omega, \mathcal{F}_{T_n})$ such that

$$0 < \phi \leq 1, \quad \phi \cdot S^{T_n} \in \mathcal{M}_{0,loc}(Q_n, \mathbb{F}).$$

Recall that for any $Q \sim P$, $\{\tilde{Z} = 1\} = \{\tilde{Z}^Q = 1\}$ where $\tilde{Z}_t^Q := Q(\tau \geq t | \mathcal{F}_t)$. Thus, a combination of this fact with $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$ leads to

$$\{\Delta(\phi \cdot S^{T_n}) \neq 0\} \cap \{\tilde{Z}^{Q_n} = 1 > Z_-^{Q_n}\} = \emptyset.$$

Therefore, by applying directly Theorem 3.12 to $\phi \cdot S^{T_n}$ under Q_n , we conclude that $\phi \cdot S^{T_n} - (\phi \cdot S^{T_n})^\tau$ (or equivalently $S^{T_n} - S^{T_n \wedge \tau}$) satisfies the NUPBR(\mathbb{G}, Q_n). Hence, the corollary follows immediately from Proposition 1.5. This ends the proof of the corollary. \square

4 Proofs of Theorems 2.1 and 3.1

In this section, we prove Theorems 2.1 and 3.1. These proofs are not technical, but are long instead.

4.1 Proof of Theorem 2.1

The proof is achieved in four steps, where we prove (c) \iff (d), (d) \iff (b), (a) \implies (c), and (b) \implies (a) respectively.

Step 1: In this step, we prove (c) \iff (d). Since S is a single jump process with predictable jump time T , then it is easy to see that S satisfies the NUPBR(R), for some probability R , is equivalent to the fact that $I_A S$ and $I_{A^c} S$ satisfies NUPBR(R) for any \mathcal{F}_{T-} -measurable event A . Hence, it is enough to prove the equivalence between assertions (d) and (c) separately on the events $\{Z_{T-} = 0\}$ and $\{Z_{T-} > 0\}$. Since $\{Z_{T-} = 0\} \subset \{\tilde{Z}_T = 0\}$ and $E(\tilde{Z}_T | \mathcal{F}_{T-}) = Z_{T-}$ on $\{T < +\infty\}$, by putting $\Gamma_0 := \{P(\tilde{Z}_T > 0 | \mathcal{F}_{T-}) = 0 \text{ \& } T < +\infty\}$, we derive

$$E(Z_{T-} I_{\Gamma_0 \cap \{T < +\infty\}}) = E(\tilde{Z}_T I_{\Gamma_0 \cap \{T < +\infty\}}) = 0,$$

and

$$\begin{aligned} 0 &= P\left(\{Z_{T-} = 0\} \cap \{\tilde{Z}_T > 0\} \cap \{T < +\infty\}\right) \\ &= E\left(I_{\{Z_{T-} = 0\} \cap \{T < +\infty\}} P(\tilde{Z}_T > 0 | \mathcal{F}_{T-})\right). \end{aligned}$$

These equalities imply that on $\{T < +\infty\}$, P -a.s., we have

$$\{Z_{T-} = 0\} = \Gamma_0 \subset \{\tilde{Z}_T = 0\}. \quad (4.36)$$

Thus, on the set $\{T < +\infty\} \cap \Gamma_0$, the three probabilities P , Q_T and \tilde{Q}_T coincide, and the equivalence between assertions (c) and (d) is obvious. On the set $\{T < +\infty \text{ \& } P[\tilde{Z}_T > 0 | \mathcal{F}_{T-}] > 0\}$, one has $\tilde{Q}_T \sim Q_T$, and the equivalence between (c) and (d) is also obvious. This achieves this first step.

Step 2: This step proves (d) \iff (b). Thanks to $\{Z_{T-} = 0\} \subset \{\tilde{Z}_T = 0\}$, we deduce that on

$\{Z_{T-} = 0\}$, $\tilde{S} \equiv S \equiv 0$ and Q_T coincides with P as well. Hence, the equivalence between assertions (d) and (b) is obvious for this case. Thus, it is enough to prove the equivalence between these assertions on $\{T < +\infty \text{ \& } P(\tilde{Z}_T > 0 | \mathcal{F}_{T-}) > 0\}$.

Assume that (d) holds. Then, there exists an \mathcal{F}_T -measurable random variable, Y , such that $Y > 0$ Q_T -a.s. and on $\{T < +\infty\}$, we have

$$E^{Q_T}(Y | \mathcal{F}_{T-}) = 1, \quad E^{Q_T}(Y |\xi| | \mathcal{F}_{T-}) < +\infty, \quad \& \quad E^{Q_T}(Y \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}) = 0.$$

Since $Y > 0$ on $\{\tilde{Z}_T > 0\}$, by putting

$$Y_1 := Y I_{\{\tilde{Z}_T > 0\}} + I_{\{\tilde{Z}_T = 0\}} \quad \text{and} \quad \tilde{Y}_1 := \frac{Y_1}{E[Y_1 | \mathcal{F}_{T-}]},$$

it is easy to check that $Y_1 > 0$, $\tilde{Y}_1 > 0$,

$$E[\tilde{Y}_1 | \mathcal{F}_{T-}] = 1 \quad \text{and} \quad E[\tilde{Y}_1 \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}] = \frac{E[Y \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}]}{E[Y_1 | \mathcal{F}_{T-}]} = 0.$$

Therefore, \tilde{S} is a martingale under $R := \tilde{Y}_1 \cdot P \sim P$, and hence \tilde{S} satisfies NUPBR(\mathbb{F}). This ends the proof of (a) \Rightarrow (b). To prove the reverse sense, we suppose that assertion (b) holds. Then, there exists $0 < Y \in L^0(\mathcal{F}_T)$, such that $E[Y |\xi| I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}] < +\infty$, $E[Y | \mathcal{F}_{T-}] = 1$ and $E[Y \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}] = 0$ on $\{Z_{T-} > 0\}$. Then, consider

$$Y_2 := \frac{Y I_{\{\tilde{Z}_T > 0\}} P(\tilde{Z}_T > 0 | \mathcal{F}_{T-})}{E[Y I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}]} + I_{\{\tilde{Z}_T = 0\}}$$

Then it is easy to verify that $Y_2 > 0$ Q_T -a.s.,

$$E^{Q_T}(Y_2 | \mathcal{F}_{T-}) = 1, \quad \text{and} \quad E^{Q_T}(Y_2 \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}) = \frac{E[Y \xi I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}]}{E[Y I_{\{\tilde{Z}_T > 0\}} | \mathcal{F}_{T-}]} = 0.$$

This proves assertion (d), and the proof of (d) \Longleftrightarrow (b) is achieved.

Step 3: Herein, we prove (a) \Rightarrow (c). Suppose that S^τ satisfies NUPBR(\mathbb{G}). Then there exists a positive \mathcal{G}_T -measurable random variable $Y^\mathbb{G}$ such that $E[\xi Y^\mathbb{G} I_{\{T \leq \tau\}} | \mathcal{G}_{T-}] = 0$ on $\{T < +\infty\}$. Due to Lemma B.2-(a), we deduce the existence of two positive \mathcal{F}_T -measurable variables $Y_1^\mathbb{F}$ and $Y_2^\mathbb{F}$ such that $Y^\mathbb{G} I_{\{T \leq \tau\}} = Y_1^\mathbb{F} I_{\{T < \tau\}} + Y_2^\mathbb{F} I_{\{T = \tau\}}$. Then, on $\{T < +\infty\}$, we obtain

$$0 = E[\xi Y^\mathbb{G} I_{\{T \leq \tau\}} | \mathcal{G}_{T-}] = E[\xi (Y_1^\mathbb{F} Z_T + (Z_T - \tilde{Z}_T) Y_2^\mathbb{F}) | \mathcal{F}_{T-}] \frac{I_{\{T \leq \tau\}}}{Z_{T-}}.$$

Therefore, by taking conditional expectation in the above equality and putting

$$\tilde{Y} := Y_1^\mathbb{F} \frac{Z_T}{\tilde{Z}_T} I_{\{\tilde{Z}_T > 0\}} + \left(\frac{Z_T}{\tilde{Z}_T} - 1\right) I_{\{\tilde{Z}_T > 0\}} Y_2^\mathbb{F} + I_{\{\tilde{Z}_T = 0\}} > 0,$$

we get

$$0 = E[\xi \tilde{Y} \frac{\tilde{Z}_T}{Z_{T-}} I_{\{Z_{T-} > 0\}} | \mathcal{F}_{T-}] = E^{\tilde{Q}^T}[\xi \tilde{Y} | \mathcal{F}_{T-}] I_{\{Z_{T-} > 0\}} = E^{\tilde{Q}^T}[S_T \tilde{Y} | \mathcal{F}_{T-}].$$

This proves that assertion (d) holds and the proof of (a) \Rightarrow (d) is achieved.

Step 4: This last step proves (b) \Rightarrow (a). Suppose that \tilde{S} satisfies NUPBR(\mathbb{F}). Then, there exists $Y \in L^1(\mathcal{F}_T)$ such that on $\{T < +\infty\}$ we have

$$E[Y|\mathcal{F}_{T-}] = 1, \quad Y > 0, \quad E[Y|\xi|I_{\{\tilde{Z}_T > 0\}}|\mathcal{F}_{T-}] < +\infty, \quad P - a.s.$$

and

$$E[Y\xi I_{\{\tilde{Z}_T > 0\}}|\mathcal{F}_{T-}] = 0.$$

Then by putting $R := Y \cdot P \sim P$, we deduce that \tilde{S} is an (\mathbb{F}, R) -martingale and $\Delta S I_{\{\tilde{Z}=0\}} \equiv 0$. As a result, assertions (a) follows from direct application of Proposition 2.8 to $M := \tilde{S}$ under $R \sim P$ (it is easy to see that (2.13) holds for (\tilde{S}, R) , i.e. $E^R(\tilde{S}_T I_{\{\tilde{Z}_T=0\}}|\mathcal{F}_{T-}) = 0$). This ends the fourth step and the proof of the theorem is completed. \square

4.2 Proof of Theorem 3.1

Due to $\{Z_{T-} = 1\} = \{P(\tilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\tilde{Z}_T = 1\}$, it is obvious that $\tilde{Q}'_T \sim Q'_T \ll P$. Thus, (b) \iff (c) follows immediately. Thus, the remaining part of the proof consists of three steps, where (c) \implies (d), (d) \implies (a) and (a) \implies (b) are proven respectively.

Step 1:(c) \Rightarrow (d). Suppose (c) holds. Then, there exists an \mathcal{F}_T -measurable random variable $Y_T > 0$, Q'_T -a.s. such that $E^{Q'_T}[S_T Y_T|\mathcal{F}_{T-}] = 0$, or equivalently

$$E[\xi Y_T I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}] I_{\{Z_{T-} < 1\}} = 0 \quad \text{and} \quad E[\xi Y_T|\mathcal{F}_{T-}] I_{\{Z_{T-} = 1\}} = 0.$$

Since, on the set $\{Z_{T-} = 1\}$, $\tilde{S} \equiv 0$, it is enough to focus on the part corresponding to $\{Z_{T-} < 1\}$. Put

$$\tilde{Y}_T := Y_T I_{\{\tilde{Z}_T < 1\}} + I_{\{\tilde{Z}_T = 1\}} \quad \text{and} \quad Q_1 := \tilde{Y}_T / E(\tilde{Y}_T|\mathcal{F}_{T-}) \cdot P \sim P.$$

Then, we derive that $E^{Q_1}[\xi I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}] = 0$. Therefore, we conclude that \tilde{S} is a (Q_1, \mathbb{F}) -martingale, and hence assertion (d) follows.

Step 2: (d) \Rightarrow (a). Since \tilde{S} satisfies NUPBR(\mathbb{F}), then there exists an \mathcal{F}_T -measurable $Y_3 > 0$ such that $E[Y_3 \xi I_{\{\tilde{Z}_T < 1\}}|\mathcal{F}_{T-}] = 0$. Put $Q_3 := Y_3 / E(Y_3|\mathcal{F}_{T-}) \cdot P \sim P$ and remark that $\{\tilde{Z}_T < 1\} = \{\tilde{Z}_T^{Q_3} < 1\}$, where $\tilde{Z}_t^{Q_3} := Q_3(\tau \geq t|\mathcal{F}_t)$. Therefore, a direct application of Theorem 3.8 under Q_3 , we conclude that $S - S^\tau = \tilde{S} - \tilde{S}^\tau$ satisfies NUPBR(\mathbb{G}).

Step 3: (a) \Rightarrow (b). Suppose $S - S^\tau$ satisfies NUPBR(\mathbb{G}). There exists a \mathcal{G}_T -measurable $Y^\mathbb{G} > 0$ such that $E[XY^\mathbb{G} I_{\{T > \tau\}}|\mathcal{G}_{T-}] = 0$. Then, thanks to Proposition ??, we deduce the existence of a positive \mathcal{F}_T -measurable $\bar{Y}^\mathbb{F}$ such that $Y^\mathbb{G} I_{\{T > \tau\}} = \bar{Y}^\mathbb{F} I_{\{T > \tau\}}$. Then, we calculate

$$\begin{aligned} 0 &= E[\xi Y^\mathbb{G} I_{\{T > \tau\}}|\mathcal{G}_{T-}] = E[\xi Y^\mathbb{F} (1 - \tilde{Z}_T)|\mathcal{F}_{T-}] \frac{I_{\{T > \tau\}}}{1 - Z_{T-}} \\ &= E^{\tilde{Q}'(T)}(XY^\mathbb{F}|\mathcal{F}_{T-}) I_{\{T > \tau\}}. \end{aligned}$$

Therefore, by taking conditional expectation and using the fact that the support of $\tilde{Q}'(T)$ is included in $\{Z_{T-} < 1\}$, we obtain

$$(1 - Z_{T-}) E^{\tilde{Q}'(T)}[\xi Y^\mathbb{F}|\mathcal{F}_{T-}] = 0, \quad \text{or equivalently} \quad E^{\tilde{Q}'(T)}[S_T Y^\mathbb{F}|\mathcal{F}_{T-}] = 0 \quad P - a.s.$$

This proves assertion (b), and the proof of the theorem is achieved. \square

5 Proof of Theorems 2.4 and 3.4

This section is devoted to the proofs of Theorems 2.1 and 3.4. These proofs are technical and require some notations on random measures and semimartingale characteristics. For any filtration \mathbb{H} , we denote

$$\tilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \quad \tilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -field on \mathbb{R}^d . To a càdlàg \mathbb{H} -adapted process X , we associate the following optional random measure μ_X defined by

$$\mu_X(dt, dx) := \sum_{u>0} I_{\{\Delta X_u \neq 0\}} \delta_{(u, \Delta X_u)}(dt, dx). \quad (5.37)$$

For a product-measurable functional $W \geq 0$ on $\Omega \times [0, +\infty[\times \mathbb{R}^d$, we denote $W \star \mu_X$ (or sometimes, with abuse of notation $W(x) \star \mu_X$) the process

$$(W \star \mu_X)_t := \int_0^t \int_{\mathbb{R}^d - \{0\}} W(u, x) \mu_X(du, dx) = \sum_{0 < u \leq t} W(u, \Delta X_u) I_{\{\Delta X_u \neq 0\}}. \quad (5.38)$$

Definitions 5.1. Consider a càdlàg \mathbb{H} -adapted process X , and its optional random measure μ_X .

(a) We denote by $\mathcal{G}_{loc}^1(\mu_X, \mathbb{H})$, the set of all $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functions, W , such that

$$\left[\sum_{t \leq \cdot} \left(W(t, \Delta S_t) I_{\{\Delta S_t \neq 0\}} - \int W_t(x) \nu_X(\{t\}, dx) \right)^2 \right]^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{H}).$$

(b) The set $\mathcal{H}_{loc}^1(\mu_X, \mathbb{H})$ is the set of all $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable functions, W , such that $(W^2 \star \mu_X)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{H})$.

Also on $\Omega \times [0, +\infty[\times \mathbb{R}^d$, we define the measure $M_{\mu_X}^P := P \otimes \mu_X$ by

$$\int W dM_{\mu_X}^P := E[(W \star \mu_X)_\infty],$$

(when the expectation is well defined). The conditional “expectation” given $\tilde{\mathcal{P}}(\mathbb{H})$ of a product-measurable functional W , denoted by $M_{\mu_X}^P(W | \tilde{\mathcal{P}}(\mathbb{H}))$, is the unique $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional \tilde{W} satisfying

$$E[(W I_\Sigma \star \mu_X)_\infty] = E[(\tilde{W} I_\Sigma \star \mu_X)_\infty], \quad \text{for all } \Sigma \in \tilde{\mathcal{P}}(\mathbb{H}).$$

When $X = S$, for the sake of simplicity, we denote $\mu := \mu_S$. Then, the \mathbb{F} -canonical decomposition of S is

$$S = S_0 + h \star (\mu - \nu) + b \cdot A + (x - h) \star \mu, \quad (5.39)$$

where h , defined as $h(x) := x I_{\{|x| \leq 1\}}$, is the truncation function. We associate to μ defined in (5.38) when $X = S$, its predictable compensator random measure ν . A direct application of Theorem A.1 in [3] (see also Theorem 3.75 in [21] (page 103), or Lemma 4.24 in [22] (Chap III)), to the martingale m defined in (2.5), leads to the existence of a local martingale m^\perp as well as $f_m \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$, $g_m \in \mathcal{H}_{loc}^1(\mu, \mathbb{F})$ and $\beta_m \in L(S^c)$ such that

$$m = \beta_m \cdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^\perp. \quad (5.40)$$

The corresponding canonical decomposition of S^τ under \mathbb{G} is given by

$$S^\tau = S_0 + h \star (\mu_b^\mathbb{G} - \nu_b^\mathbb{G}) + h \frac{f_m}{Z_-} I_{[0, \tau]} \star \nu + b \bullet A^\tau + (x - h) \star \mu_b^\mathbb{G} \quad (5.41)$$

where (β_m, f_m) is given by (5.40) and $\mu_b^\mathbb{G}$ and $\nu_b^\mathbb{G}$ are given by

$$\mu_b^\mathbb{G}(dt, dx) := I_{[0, \tau]}(t) \mu(dt, dx), \quad \nu_b^\mathbb{G}(dt, dx) := (1 + Z_-^{-1} f_m) I_{[0, \tau]}(t) \nu(dt, dx). \quad (5.42)$$

5.1 Proof of Theorem 2.4

This proof consists of four steps, where we prove (b) \iff (c), (b) \implies (a), and (a) \implies (b) respectively. Only the last step is technically involved.

Step 1: Here, we prove (b) \iff (c). Remark that (c) \implies (b) follows immediately from Lemma 1.4. Suppose that assertion (b) holds, and consider the following \mathbb{F} -predictable process

$$\varphi := \left[1 + {}^{p, \mathbb{F}} \left(Y | \Delta S | I_{\{\tilde{Z} > 0\}} \right) \right]^{-1} \left[I_{\Omega \setminus (\cup_n [T_n])} + \sum 2^{-n} I_{[T_n]} \right],$$

where $(T_n)_n$ a sequence of \mathbb{F} -predictable stopping times such that $\{\Delta S \neq 0\} \subset \bigcup_{n=1}^{+\infty} [T_n]$. Then, it is easy to see that the process

$$X := Y_- \varphi \cdot S^{(0)} + [\varphi \cdot S^{(0)}, Y] = \sum Y \varphi \Delta S I_{\{\tilde{Z} < 1 \text{ \& } Z_- \geq \delta\}}$$

has an integrable variation and its \mathbb{F} -compensator is given by (due to the fact it is a pure jump process with finite variation and it jumps on predictable stopping times only)

$$X^{p, \mathbb{F}} = \sum {}^{p, \mathbb{F}} \left(Y \varphi \Delta S I_{\{\tilde{Z} > 0\}} \right) I_{\{Z_- \geq \delta\}} \equiv 0.$$

Thus, $Y(\varphi \cdot S^{(0)})$ is an \mathbb{F} -local martingale, and $S^{(0)}$ satisfies the NUPBR(\mathbb{F}). This ends the proof of (b) \iff (c).

Step 2: Here, we prove (b) \implies (a). Suppose that assertion (b) holds, and consider a sequence of \mathbb{F} -stopping times $(\tau_n)_n$ that increases to infinity such that Y^{τ_n} is an \mathbb{F} -martingale, and $Q_n := Y_{\tau_n}/Y_0 \cdot P \sim P$. Then, (2.9) implies that $(S^{(0)})^{\sigma_n}$ is a Q_n -local martingale and satisfies (2.19) under Q_n due to

$$\{\tilde{Z}_T^Q = 0\} = \{\tilde{Z}_T = 0\}, \text{ for any } Q \sim P \text{ and any } \mathbb{F}\text{-stopping time } T, \quad (5.43)$$

where $\tilde{Z}_t^Q := Q[\tau \geq t | \mathcal{F}_t]$. This follows from

$$E \left[\tilde{Z}_T I_{\{\tilde{Z}_T^Q = 0\}} \right] = E \left[I_{\{\tau \geq T\}} I_{\{\tilde{Z}_T^Q = 0\}} \right] = 0,$$

(which implies $\{\tilde{Z}_T^Q = 0\} \subset \{\tilde{Z}_T = 0\}$) and the symmetric role of Q and P .

Thus, a direct application of Theorem 2.12 to $((S^{(0)})^{\sigma_n}, Q_n)$ leads to the NUPBR(\mathbb{G}, Q_n) of $(S^{(0)})^{\sigma_n \wedge \tau} = (I_{\{Z_- \geq \delta\}} \cdot S)^{\sigma_n \wedge \tau}$. Thanks to Proposition 1.5, this implies the NUPBR(\mathbb{G}) of $I_{\{Z_- \geq \delta\}} \cdot S$ for any $\delta > 0$. Since $Z_-^{-1} I_{[0, \tau]}$ is \mathbb{G} -locally bounded, there exists a family of \mathbb{G} -stopping times τ_δ that increases to infinity when δ decreases to zero, and $[0, \tau \wedge \tau_\delta] \subset \{Z_- \geq \delta\}$. Therefore, we conclude that $S^{\tau \wedge \tau_\delta}$ satisfies the NUPBR(\mathbb{G}). Hence, again Proposition 1.5 implies finally that S^τ satisfies the NUPBR(\mathbb{G}). This ends the second part.

Step 3: In this step, we focus on proving (a) \implies (b). Suppose that S^τ satisfies NUPBR(\mathbb{G}). Then,

there exists a σ -martingale density under \mathbb{G} , for $I_{\{Z_- \geq \delta\}} \cdot S^\tau$, ($\delta > 0$), that we denote by $D^\mathbb{G}$. Then, from a direct application of Theorem A.1, we deduce the existence of a positive $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional, $f^\mathbb{G} \in \mathcal{G}_{loc}^1(\mu_b^\mathbb{G}, \mathbb{G})$, such that $D^\mathbb{G} := \mathcal{E}(N^\mathbb{G}) > 0$, with

$$N^\mathbb{G} := W^\mathbb{G} \star (\mu^\mathbb{G} - \nu^\mathbb{G}), \quad W^\mathbb{G} := f^\mathbb{G} - 1 + \frac{\widehat{f}^\mathbb{G} - a^\mathbb{G}}{1 - a^\mathbb{G}} I_{\{a^\mathbb{G} < 1\}},$$

where $\nu^\mathbb{G}$ was defined in (5.42), and, introducing f_m defined in (5.40)

$$xf^\mathbb{G} I_{\{Z_- \geq \delta\}} \star \nu^\mathbb{G} = xf^\mathbb{G} \left(1 + \frac{f_m}{Z_-}\right) I_{\llbracket 0, \tau \rrbracket} I_{\{Z_- \geq \delta\}} \star \nu \equiv 0. \quad (5.44)$$

Thanks to Lemma B.2, we conclude the existence of a positive $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional, f , such that $f^\mathbb{G} I_{\llbracket 0, \tau \rrbracket} = f I_{\llbracket 0, \tau \rrbracket}$. Thus, (5.44) becomes

$$U^{(b)} := xf \left(1 + \frac{f_m}{Z_-}\right) I_{\llbracket 0, \tau \rrbracket} I_{\{Z_- > 0\}} \star \nu \equiv 0.$$

Introduce the following notations

$$\begin{cases} \mu_0 := I_{\{\tilde{Z} > 0 \text{ \& } Z_- \geq \delta\}} \cdot \mu, \quad \nu_0 := h_0 I_{\{Z_- \geq \delta\}} \cdot \nu, \quad h_0 := M_\mu^P \left(I_{\{\tilde{Z} > 0\}} | \tilde{\mathcal{P}} \right), \\ g := \frac{f(1 + \frac{f_m}{Z_-})}{h_0} I_{\{h_0 > 0\}} + I_{\{h_0 = 0\}}, \quad a_0(t) := \nu_0(\{t\}, \mathbb{R}^d), \end{cases} \quad (5.45)$$

and assume that

$$\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}_{loc}^+(\mathbb{F}). \quad (5.46)$$

Then, thanks to Lemma A.2, we deduce that $W := (g-1)/(1-a^0 + \widehat{g}) \in \mathcal{G}_{loc}^1(\mu_0, \mathbb{F})$, and the local martingales

$$N^{(0)} := \frac{g-1}{1-a^0 + \widehat{g}} \star (\mu_0 - \nu_0), \quad Y^{(0)} := \mathcal{E}(N^{(0)}), \quad (5.47)$$

are well defined satisfying $1 + \Delta N^{(0)} > 0$, $[N^{(0)}, S] \in \mathcal{A}(\mathbb{F})$, and on $\{Z_- > 0\}$ we have

$$\begin{aligned} \frac{{}^{p, \mathbb{F}} \left(Y^{(0)} \Delta S I_{\{\tilde{Z} > 0\}} \right)}{Y_-^{(0)}} &= {}^{p, \mathbb{F}} \left((1 + \Delta N^{(0)}) \Delta S I_{\{\tilde{Z} > 0\}} \right) = {}^{p, \mathbb{F}} \left(\frac{g}{1-a^0 + \widehat{g}} \Delta S I_{\{\tilde{Z} > 0\}} \right) \\ &= \Delta \frac{gxh_0}{1-a^0 + \widehat{g}} \star \nu = \Delta \frac{xf(1 + \frac{f_m}{Z_-})}{1-a^0 + \widehat{g}} I_{\{Z_- > 0\}} \star \nu \\ &= \frac{{}^{p, \mathbb{F}} (\Delta U^{(b)})}{1-a^0 + \widehat{g}} \equiv 0. \end{aligned}$$

This proves that assertion (b) holds under the assumption (5.46). The remaining part of the proof will show that this assumption holds always. To this end, we start by noticing that on the set $\{h_0 > 0\}$,

$$\begin{aligned} g-1 &= \frac{f(1 + \frac{f_m}{Z_-})}{h_0} - 1 = \frac{(f-1)(1 + \frac{f_m}{Z_-})}{h_0} + \frac{f_m}{Z_- h_0} + \frac{M_\mu^P \left(I_{\{\tilde{Z}=0\}} | \tilde{\mathcal{P}} \right)}{h_0} \\ &=: \frac{(f-1)(1 + \frac{f_m}{Z_-})}{h_0} + \frac{M_\mu^P \left(\Delta m I_{\{\tilde{Z} > 0\}} | \tilde{\mathcal{P}} \right)}{Z_- h_0} =: g_1 + \frac{M_\mu^P \left(\Delta m I_{\{\tilde{Z} > 0\}} | \tilde{\mathcal{P}} \right)}{Z_- h_0}. \end{aligned}$$

Since $((f-1)^2 I_{[0,\tau]} \star \mu)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{G})$, then due to Proposition A.3-(e)

$$\sqrt{(f-1)^2 I_{\{Z_- \geq \delta\}} \star (\tilde{Z} \cdot \mu)} \in \mathcal{A}_{loc}^+(\mathbb{F}), \quad \text{for any } \delta > 0.$$

Then, a direct application of Proposition A.3-(a), for any $\delta > 0$, we have

$$(f-1)^2 I_{\{|f-1| \leq \alpha \ \& \ Z_- \geq \delta\}} \star (\tilde{Z} \cdot \mu), \quad |f-1| I_{\{|f-1| > \alpha \ \& \ Z_- \geq \delta\}} \star (\tilde{Z} \cdot \mu) \in \mathcal{A}_{loc}^+(\mathbb{F}).$$

By stopping, without loss of generality, we assume these two processes and $[m, m]$ belong to $\mathcal{A}^+(\mathbb{F})$. Remark that $Z_- + f_m = M_\mu^P(\tilde{Z}|\tilde{\mathcal{P}}) \leq M_\mu^P(I_{\{\tilde{Z} > 0\}}|\tilde{\mathcal{P}}) = h_0$ that follows from $\tilde{Z} \leq I_{\{\tilde{Z} > 0\}}$. Therefore, we derive

$$\begin{aligned} E[g_1^2 I_{\{|f-1| \leq \alpha\}} \star \mu_0(\infty)] &= E\left[\frac{(f-1)^2(1 + \frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1| \leq \alpha\}} \star \mu_0(\infty)\right] \\ &= E\left[\frac{(f-1)^2(1 + \frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1| \leq \alpha\}} \star \nu_0(\infty)\right] \\ &\leq \delta^{-2} E[(f-1)^2(Z_- + f_m) I_{\{|f-1| \leq \alpha \ \& \ Z_- \geq \delta\}} \star \nu(\infty)] \\ &= \delta^{-2} E[(f-1)^2 I_{\{|f-1| \leq \alpha\}} \star (\tilde{Z} I_{\{Z_- \geq \delta\}} \cdot \mu)(\infty)] < +\infty, \end{aligned}$$

and

$$\begin{aligned} E[g_1 I_{\{|f-1| > \alpha\}} \star \mu_0(\infty)] &= E\left[\frac{|f-1|(1 + \frac{f_m}{Z_-})}{h_0} I_{\{|f-1| > \alpha\}} \star \mu_0(\infty)\right] \\ &= E\left[|f-1|(1 + \frac{f_m}{Z_-}) I_{\{|f-1| > \alpha\}} I_{\{Z_- \geq \delta\}} \star \nu_0(\infty)\right] \\ &\leq \delta^{-1} E[|f-1| I_{\{|f-1| > \alpha\}} \star (\tilde{Z} I_{\{Z_- \geq \delta\}} \cdot \mu)(\infty)] < +\infty. \end{aligned}$$

Here μ_0 and ν_0 are defined in (5.45). Therefore, again by Proposition A.3-(a), we conclude that $\sqrt{g_1^2 \star \mu_0} \in \mathcal{A}_{loc}^+(\mathbb{F})$.

Due to $M_\mu^P(HK|\tilde{\mathcal{P}}(\mathbb{F}))^2 \leq M_\mu^P(H^2|\tilde{\mathcal{P}}(\mathbb{F}))M_\mu^P(K^2|\tilde{\mathcal{P}}(\mathbb{F}))$, we derive

$$\begin{aligned} E\left[\frac{M_\mu^P(\Delta m I_{\{\tilde{Z} > 0\}}|\tilde{\mathcal{P}})^2}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] &\leq E\left[\frac{M_\mu^P((\Delta m)^2|\tilde{\mathcal{P}}) M_\mu^P(I_{\{\tilde{Z} > 0\}}|\tilde{\mathcal{P}})}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\ &= E\left[\frac{M_\mu^P((\Delta m)^2|\tilde{\mathcal{P}})}{Z_-^2} I_{\{Z_- \geq \delta\}} \star \mu(\infty)\right] \\ &\leq \delta^{-2} E[[m, m]_\infty] < +\infty. \end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}_{loc}^+(\mathbb{F})$. This ends the proof of (5.46), and the proof of the theorem is completed. \square

5.2 Proof of Theorem 3.4

Before proving the equivalence between the two assertions of the theorem, we will start outlining a number of remarks that simplify tremendously the proof. It is easy to prove that on $\{T < +\infty\}$ we have

$$\{\tilde{Z}_T^Q = 1\} = \{\tilde{Z}_T = 1\} \quad \text{for } Q \sim P \text{ and } \mathbb{F}\text{-stopping time } T, \quad (5.48)$$

where $\tilde{Z}_t^Q := E^Q(\tau \geq t | \mathcal{F}_t)$. Indeed, due to

$$E \left[(1 - \tilde{Z}_T) I_{\{\tilde{Z}_T^Q = 1\}} \right] = E \left[I_{\{\tau < T\}} I_{\{\tilde{Z}_T^Q = 1\}} \right] = 0,$$

the inclusion $\{\tilde{Z}_T^Q = 1\} \subset \{\tilde{Z}_T = 1\}$ follows, while the reverse inclusion follows by symmetry. This proves (5.48).

Since S is a thin process with predictable jump times only, then there exists a sequence of \mathbb{F} -predictable stopping times, $(T_n)_{n \geq 1}$, such that

$$\{\Delta S \neq 0\} \subset \bigcup_{n=1}^{+\infty} \llbracket T_n \rrbracket.$$

The proof of the theorem consists of three steps in which we prove (b) \iff (c), (b) \implies (a) and (a) \implies (b) respectively.

Step 1: Here, we prove (b) \iff (c). Remark that, thanks to Lemma 1.4, (c) \implies (b) follows immediately. To prove the reverse (i.e. (b) \implies (c)), we consider the following \mathbb{F} -predictable process

$$\varphi := \left[1 + {}^{p, \mathbb{F}} \left(Y | \Delta S | I_{\{\tilde{Z} < 1\}} \right) \right]^{-1} \left[I_{\Omega \setminus (\bigcup_{n=1}^{+\infty} \llbracket T_n \rrbracket)} + \sum_{n=1}^{+\infty} 2^{-n} I_{\llbracket T_n \rrbracket} \right].$$

It is easy to check that $0 < \varphi \leq 1$ and $U := Y_- \varphi \cdot S^{(1)} + [Y, \varphi \cdot S^{(1)}]$ is a process with integrable variation whose compensator (since it is a pure jump process with finite variation and jumps on predictable stopping times only) is

$$U^{p, \mathbb{F}} = \sum \varphi {}^{p, \mathbb{F}} \left(Y \Delta S I_{\{\tilde{Z}=1 > Z_- \}} \right) \equiv 0.$$

This proves that Y is σ -martingale density for $S^{(1)}$ (i.e. $Y \in \mathcal{L}(S^{(1)}, \mathbb{F})$), and hence assertion (c) follows immediately.

Step 2: Here we will prove (b) \implies (a). Suppose that assertion (b) holds, and consider a sequence of \mathbb{F} -stopping times $(\sigma_n)_n$ such that Y^{σ_n} is a martingale, and put $Q_n := (Y_{\sigma_n}/Y_0) \cdot P \sim P$. Then, since $\tilde{S} := \sum \Delta S I_{\{\tilde{Z} < 1\}}$ is a thin process with predictable jump times only, the condition (3.24) translates into the fact that \tilde{S}^{σ_n} is a Q_n -local martingale satisfying

$$\{\Delta \tilde{S}^{\sigma_n} \neq 0\} \cap \{\tilde{Z}^{Q_n} = 1 > Z_-^{Q_n}\} = \emptyset,$$

due to (5.48). Therefore, thanks to Proposition 1.5, it is enough to prove that assertion (a) holds true under Q_n for S^{σ_n} . Therefore, without loss of generality, we assume $Y \equiv 1$ and hence \tilde{S} is a \mathbb{F} -local martingale satisfying (3.34). Thus, a direct application of Theorem 3.12 implies that \tilde{S}^τ satisfies the NUPBR(\mathbb{G}).

Step 3: Here, we will prove (a) \implies (b). Suppose that $S - S^\tau$ satisfies the NUPBR(\mathbb{G}). A direct application Theorem A.1 implies the existence of $f^\mathbb{G} \in \mathcal{G}_{loc}^1(\mu_a^\mathbb{G}, \mathbb{G})$ such that $f^\mathbb{G} > 0$,

$$N^\mathbb{G} := W^\mathbb{G} \star (\mu_a^\mathbb{G} - \nu_a^\mathbb{G}), \quad W^\mathbb{G} := f^\mathbb{G} - 1 + \frac{\widehat{f}^\mathbb{G} - a^\mathbb{G}}{1 - a^\mathbb{G}} I_{\{a^\mathbb{G} < 1\}},$$

and

$$xf^{\mathbb{G}} \star \nu_a^{\mathbb{G}} = xf^{\mathbb{G}} \left(1 - \frac{f_m}{1 - Z_-} \right) I_{\llbracket \tau, +\infty \rrbracket} \star \nu \equiv 0. \quad (5.49)$$

Here $f_m := M_\mu^P(\Delta m | \tilde{\mathcal{P}}(\mathbb{F}))$ (given also by (5.40), and $\mu_a^{\mathbb{G}}$ and $\nu_a^{\mathbb{G}}$ are given by

$$\mu_a^{\mathbb{G}} := I_{\llbracket \tau, +\infty \rrbracket} \cdot \mu, \quad \nu_a^{\mathbb{G}} := \left(1 - \frac{f_m}{1 - Z_-} \right) I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu.$$

Thanks to Lemma B.2, there exists an $\mathcal{P}(\mathbb{F})$ -measure functional $f > 0$ such that $f^{\mathbb{G}} = f$ on the stochastic interval $\llbracket \tau, +\infty \rrbracket$, and (5.49) becomes

$$xf \left(1 - \frac{f_m}{1 - Z_-} \right) I_{\llbracket \tau, +\infty \rrbracket} \star \nu \equiv 0. \quad (5.50)$$

Due to Proposition A.4 and \mathbb{G} -locally boundedness of $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$, we could find a sequence of \mathbb{F} -stopping time $(\sigma_n^{\mathbb{F}})_{n \geq 1}$ that increases to infinity and $(1 - Z_-)^{-1} I_{\llbracket 0, \sigma_n^{\mathbb{F}} \rrbracket} I_{\llbracket \tau, +\infty \rrbracket}$ is bounded by $(n + 1)$. Also, since $((f - 1)^2 I_{\llbracket \tau, +\infty \rrbracket} \star \mu)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{G})$, thanks to Proposition A.4 (both assertions (c) and (a)) we deduce the existence of a sequence of \mathbb{F} -stopping times $(\tau_n)_n$ that increases to infinity such that the three processes $[m, m]^{\tau_n}$

$$(f - 1)^2 I_{\{|f-1| \leq \alpha \text{ \& } 1-Z_- \geq \delta\}} \star \bar{\mu}^{\tau_n} \text{ and } |f - 1| I_{\{|f-1| > \alpha \text{ \& } 1-Z_- \geq \delta\}} \star \bar{\mu}^{\tau_n}$$

are integrable, where $\bar{\mu} := (1 - \tilde{Z}) \cdot \mu$. Consider the following notations

$$\begin{aligned} \mu_1 &:= I_{\{\tilde{Z} < 1 \text{ \& } 1-Z_- \geq \delta\}} \cdot \mu, \quad \nu_1 := h_1 I_{\{1-Z_- \geq \delta\}} \cdot \nu, \quad h_1 := M_\mu^P \left(I_{\{\tilde{Z} < 1\}} | \tilde{\mathcal{P}} \right), \\ g &:= \frac{f(1 - \frac{f_m}{1-Z_-})}{h_1} I_{\{h_1 > 0 \text{ \& } Z_- < 1\}} + I_{\{h_1 = 0 \text{ or } Z_- = 1\}}, \end{aligned}$$

and suppose that

$$W^{(1)}(t, x) := \frac{g_t(x) - 1}{1 - a_t^{(1)} + \hat{g}_t} \in \mathcal{G}_{loc}^1(\mu_1, \mathbb{F}), \quad (5.51)$$

where $a_t^{(1)} := \nu_1(\{t\}, \mathbb{R}^d)$ and $\hat{g}_t := \int g_t(x) \nu_1(\{t\}, dx)$.

Then, we can easily prove that assertion (b) holds. In fact, we take

$$N^{(1)} := \frac{g - 1}{1 - a^{(1)} + \hat{g}} \star (\mu_1 - \nu_1) \quad \text{and} \quad Y := \mathcal{E}(N^{(1)}).$$

Then, it is clear that

$$1 + \Delta N^{(1)} = \frac{1}{1 - a^{(1)} + \hat{g}} I_{\{\Delta S = 0 \text{ or } \tilde{Z} = 1\}} + \frac{g(\Delta S)}{1 - a^{(1)} + \hat{g}} I_{\{\Delta S \neq 0 \text{ \& } \tilde{Z} < 1\}} > 0,$$

and on $\{Z_- < 1\}$ we get

$$\begin{aligned} {}^{p, \mathbb{F}} \left(Y \Delta S I_{\{\tilde{Z} < 1\}} \right)_t &= Y_{t-} {}^{p, \mathbb{F}} \left((1 + \Delta N^{(1)}) \Delta S I_{\{\tilde{Z} < 1\}} \right)_t = \frac{Y_{t-} {}^{p, \mathbb{F}} \left(g(\Delta S) \Delta S I_{\{\tilde{Z} < 1\}} \right)_t}{1 - a_t^{(1)} + \hat{g}_t} \\ &= \frac{Y_{t-}}{1 - a_t^{(1)} + \hat{g}_t} \int g_t(x) x h_1(t, x) \nu(\{t\}, dx) \\ &= \frac{Y_{t-}}{1 - a_t^{(1)} + \hat{g}_t} \int x f_t(x) \left(1 - \frac{f_m(t, x)}{1 - Z_-} \right) \nu(\{t\}, dx) \equiv 0. \end{aligned}$$

The last equality in the above string of equalities follows direct from (5.50). Therefore, assertion (b) will follow immediately as long as we prove (5.51). To this end, on $\{h_1 > 0 \text{ \& } Z_- < 1\}$ we calculate

$$\begin{aligned} g - 1 &= \frac{f(1 - Z_- - f_m)}{h_1(1 - Z_-)} - 1 = \frac{(f - 1)(1 - Z_- - f_m)}{h_1(1 - Z_-)} - \frac{M_\mu^P \left(\Delta m I_{\{\tilde{Z} < 1\}} | \tilde{\mathcal{P}} \right)}{(1 - Z_-)h_1} \\ &=: g_1 + g_2, \end{aligned}$$

and remark that $\{1 - Z_- - f_m > 0\} \subset \{h_0 > 0\}$ which is due to

$$1 - Z_- - f_m = 1 - M_\mu^P \left(\tilde{Z} | \tilde{\mathcal{P}} \right) \leq M_\mu^P \left(I_{\{\tilde{Z} < 1\}} | \tilde{\mathcal{P}} \right) = h_1,$$

that is implied by $I_{\{\tilde{Z}=1\}} \leq \tilde{Z}$. Therefore, we derive that

$$\begin{aligned} E \left[g_1^2 I_{\{|f-1| \leq \alpha\}} \star \mu_0(\sigma_n \wedge \tau_n) \right] &= E \left[\frac{(f-1)^2(1 - Z_- - f_m)^2}{h_1^2(1 - Z_-)^2} I_{\{|f-1| \leq \alpha\}} I_{\{\tilde{Z} < 1\}} \star \mu(\sigma_n \wedge \tau_n) \right] \\ &= E \left[\frac{(f-1)^2(1 - Z_- - f_m)^2}{h_1(1 - Z_-)^2} I_{\{|f-1| \leq \alpha\}} \star \nu(\sigma_n \wedge \tau_n) \right] \\ &\leq E \left[(f-1)^2 \frac{1 - Z_- - f_m}{(1 - Z_-)^2} I_{\{|f-1| \leq \alpha\}} \star \nu(\sigma_n \wedge \tau_n) \right] \\ &\leq E \left[(f-1)^2 \frac{1}{(1 - Z_-)^2} I_{\{|f-1| \leq \alpha\}} \star ((1 - \tilde{Z}) \cdot \mu(\sigma_n \wedge \tau_n)) \right] \\ &= E \left[(f-1)^2 \frac{1}{(1 - Z_-)^2} I_{\{|f-1| \leq \alpha\}} I_{\tau, +\infty} \star \mu(\sigma_n \wedge \tau_n) \right] \\ &\leq (n+1)^2 E \left[(f-1)^2 I_{\{|f-1| \leq \alpha\}} \star \mu(\tau_n) \right] < +\infty. \end{aligned}$$

and

$$\begin{aligned} E \left[g_1 I_{\{|f-1| > \alpha\}} \star \mu_1(\sigma_n \wedge \tau_n) \right] &= E \left[\frac{|f-1|(1 - Z_- - f_m)}{h_1(1 - Z_-)} I_{\{|f-1| > \alpha\}} I_{\{\tilde{Z} < 1\}} \star \mu(\sigma_n \wedge \tau_n) \right] \\ &= E \left[|f-1| \frac{1 - Z_- - f_m}{1 - Z_-} I_{\{|f-1| > \alpha\}} \star \nu(\sigma_n \wedge \tau_n) \right] \\ &\leq E \left[|f-1| \frac{1}{1 - Z_-} I_{\{|f-1| > \alpha\}} \star ((1 - \tilde{Z}) \cdot \mu)(\sigma_n \wedge \tau_n) \right] \\ &= E \left[|f-1| \frac{1}{1 - Z_-} I_{\{|f-1| > \alpha\}} I_{\tau, +\infty} \star \mu(\sigma_n \wedge \tau_n) \right] \\ &\leq (n+1) E \left[|f-1| I_{\{|f-1| > \alpha\}} \star ((1 - \tilde{Z}) \cdot \mu)(\tau_n) \right] < +\infty. \end{aligned}$$

This proves that $\sqrt{g_1^2 \star \mu_1} \leq \sqrt{2} \sqrt{g_1^2 I_{\{|f-1| > \alpha\}} \star \mu_1} + \sqrt{2} (g_1 I_{\{|f-1| > \alpha\}} \star \mu_1)$ belongs to $\mathcal{A}_{loc}^+(\mathbb{F})$. To prove $\sqrt{g_2^2 \star \mu_1} \in \mathcal{A}_{loc}^+(\mathbb{F})$, we derive

$$\begin{aligned} E \left[(g_2)^2 \star \mu_0(\sigma_n \wedge \tau_n) \right] &= E \left[\frac{M_\mu^P \left(\Delta m I_{\{\tilde{Z} < 1\}} | \tilde{\mathcal{P}} \right)^2}{(1 - Z_-)^2 h_0^2} I_{\{\tilde{Z} < 1 \text{ \& } Z_- < 1\}} \star \mu(\sigma_n \wedge \tau_n) \right] \\ &\leq E \left[\frac{M_\mu^P \left((\Delta m)^2 | \tilde{\mathcal{P}} \right) M_\mu^P \left(I_{\{\tilde{Z} < 1 \text{ \& } Z_- < 1\}} | \tilde{\mathcal{P}} \right)^2}{(1 - Z_-)^2 h_1^2} \star \nu(\sigma_n \wedge \tau_n) \right] \end{aligned}$$

$$\begin{aligned}
&= E \left[\frac{M_\mu^P \left((\Delta m)^2 | \tilde{\mathcal{P}} \right)}{(1 - Z_-)^2} I_{\{Z_- < 1\}} \star \nu(\sigma_n \wedge \tau_n) \right] \\
&\leq E \left[\frac{1}{(1 - Z_-)^3} I_{\tau, +\infty} \cdot \left(M_\mu^P \left((\Delta m)^2 | \tilde{\mathcal{P}} \right) \star \nu \right)_{\sigma_n \wedge \tau_n} \right] \\
&\leq (n+1)^3 E([m, m]_{\tau_n}) < +\infty.
\end{aligned}$$

Hence, $\sqrt{(g-1)^2 \star \mu_1} \in \mathcal{A}_{loc}^+(\mathbb{F})$ follows. Thanks to Lemma A.2 (see Choulli and Schweizer(2012) [10]), (5.51) follows immediately. This ends the proof of the theorem. \square

A Integrality Results

Theorem A.1. *Let S be a semi-martingale with predictable characteristic triplet $(b, c, \nu = A \otimes F)$, N is a local martingale such that $\mathcal{E}(N) > 0$, and (β, f, g, N') are its Jacod's parameters. Then the following assertions hold.*

1) $\mathcal{E}(N)$ is a σ -martingale density of S if and only if the following two properties hold:

$$\int |x - h(x) + xf(x)| F(dx) < +\infty, \quad P \otimes A - a.e. \quad (\text{A.52})$$

and

$$b + c\beta + \int (x - h(x) + xf(x)) F(dx) = 0, \quad P \otimes A - a.e. \quad (\text{A.53})$$

2) In particular, we have

$$\int x(1 + f_t(x)\nu(\{t\}, dx) = \int x(1 + f_t(x)F_t(dx))\Delta A_t = 0, \quad P - a.e. \quad (\text{A.54})$$

Proof. The proof can be found in Choulli et al. [9, Lemma 2.4] 2007, and also Choulli and Schweizer (2013) [10]. \square

Lemma A.2. *Let f be a $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional such that $f > 0$ and*

$$\left[(f-1)^2 \star \mu \right]^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{H}). \quad (\text{A.55})$$

Then, the \mathbb{H} -predictable process $(1 - a^{\mathbb{H}} + \hat{f}^{\mathbb{H}})^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a_t^{\mathbb{H}} + \hat{f}_t^{\mathbb{H}}} \in \mathcal{G}_{loc}^1(\mu, \mathbb{H}). \quad (\text{A.56})$$

Here, $a_t^{\mathbb{H}} := \nu^{\mathbb{H}}(\{t\}, \mathbb{R}^d)$, $\hat{f}_t^{\mathbb{H}} := \int f_t(x)\nu^{\mathbb{H}}(\{t\}, dx)$ and $\nu^{\mathbb{H}}$ is the \mathbb{H} -predictable random measure compensator of μ under \mathbb{H} .

Proof. The proof of this lemma can be found in Choulli and Schweizer (2013). Below we provide this proof for the sake of completeness. In this proof we put

$$U_t(x) = 1 - f_t(x), \quad \text{and} \quad \hat{U}_t := a_t^{\mathbb{H}} - \hat{f}_t^{\mathbb{H}}.$$

We start by remarking that (A.56) follows from the combination of (A.55) and the local boundedness of $1/(1 - \widehat{U})$. Therefore, in what follows, we will focus on proving this latter fact. Consider $\delta \in (0, 1)$, $\eta \in (0, 1)$, and the stopping times and processes defined by

$$T_0 = 0, \quad T_{n+1} := \inf \left\{ t > T_n \mid \sum_{T_n < v \leq t} (U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}})^2 > \delta^2 \right\},$$

$$V_n(t) := \left[\sum_{T_n < v \leq t} (U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}})^2 \right]^{1/2}$$

Remark that —since for each $n \geq 0$, the process $(V_n(t))^2$ is RCLL and nondecreasing real-valued process— we have

$$(V_n(T_{n+1}))^2 := \sum_{T_n < v \leq T_{n+1}} (U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}})^2 \geq \delta^2 \quad \text{on } \{T_{n+1} < +\infty\}.$$

This implies that T_n increases to $+\infty$ almost surely, and

$$V_n(t-) \leq \delta, \quad P - a.s. \quad \text{for all } t \leq T_{n+1}.$$

Due to $0 \leq (1 - \widehat{U})^{-1} I_{\{\widehat{U} < 1 - \eta\}} \leq \eta^{-1}$ and

$$(1 - \widehat{U})^{-1} = (1 - \widehat{U})^{-1} I_{\{\widehat{U} \geq 1 - \eta\}} + (1 - \widehat{U})^{-1} I_{\{\widehat{U} < 1 - \eta\}},$$

we deduce that the proof of the lemma will be achieved once we prove that

$$Y := \frac{1}{1 - \widehat{U}} I_{\{\widehat{U} \geq 1 - \eta\}}$$

is locally bounded. Thanks to Dellacherie and Meyer (1980), this fact is equivalent to

$$\sup_{0 \leq u \leq t} Y_u < +\infty \quad P - a.s. \quad \text{for any } t \in (0, +\infty).$$

Since T_n increases to ∞ almost surely, then this fact is implied by

$$\sup_{T_n \leq u \leq t \wedge T_{n+1}} Y_u < +\infty \quad P - a.s. \quad \text{on } \{t > T_n\}.$$

Simple calculation leads to

$$\widehat{U}_s \leq V_n(s-) + {}^{p, \mathbb{H}}(\Delta V_n)_s, \quad \text{for all } T_n < s \leq T_{n+1}.$$

Thus, it is easy to see that for $\delta + \eta < 1$,

$$\begin{aligned} \{s \in]T_n, T_{n+1}] \mid \widehat{U}_s \geq 1 - \eta\} &\subset \{s \in]T_n, T_{n+1}] \mid {}^{p, \mathbb{H}}(\Delta V_n)_s \geq 1 - \eta - V_n(s-)\} \\ &\subset \{s \in]T_n, T_{n+1}] \mid \Delta((V_n)^{p, \mathbb{H}}) = {}^{p, \mathbb{H}}(\Delta V_n)_s \geq 1 - \eta - \delta\} =: \Gamma_n. \end{aligned}$$

It is obvious that $\#(\Gamma_n \cap [0, t]) < +\infty$ $P - a.s.$ since $(V_n)^{p, \mathbb{H}}$ is a càdlàg process. Thus, we deduce that

$$\sup_{T_n \leq u \leq t \wedge T_{n+1}} Y_u = \max_{T_n \leq u \leq t \wedge T_{n+1}} Y_u < +\infty.$$

This ends the proof of the lemma. \square

\square

Proposition A.3. *For any $\alpha > 0$, the following assertions hold:*

(a) *Let h be a $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional. Then, $\sqrt{(h-1)^2} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{H})$ iff*

$$(h-1)^2 I_{\{|h-1| \leq \alpha\}} \star \mu \text{ and } |h-1| I_{\{|h-1| > \alpha\}} \star \mu \text{ belong to } \mathcal{A}_{loc}^+(\mathbb{H}).$$

(b) *Let V be an \mathbb{F} -predictable and non-decreasing process. Then, $V^\tau \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if $I_{\{Z_- \geq \delta\}} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ for any $\delta > 0$.*

(c) *Let h be a nonnegative and $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional. Then, $h I_{\llbracket 0, \tau \rrbracket} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if for all $\delta > 0$, $h I_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}_{loc}^+(\mathbb{F})$, where $\mu^1 := \tilde{Z} \cdot \mu$.*

(d) *Let f be positive and $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable, and $\mu^1 := \tilde{Z} \cdot \mu$. Then $\sqrt{(f-1)^2} I_{\llbracket 0, \tau \rrbracket} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ iff $\sqrt{(f-1)^2} I_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}_{loc}^+(\mathbb{F})$, for all $\delta > 0$.*

Proposition A.4. *Suppose that τ is a finite honest time satisfying (3.20). Then, the following properties hold.*

(a) *Let $\Phi^\mathbb{G}$ a \mathbb{G} -predictable process and k a nonnegative and $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional such that $0 < \Phi^\mathbb{G} \leq 1$ and $\Phi^\mathbb{G} k \star \mu_\mathbb{G} \in \mathcal{A}_{loc}^+(\mathbb{G})$. Then, $P \otimes A$ -a.e.*

$$\int k(x) (1 - Z_- - f_m(x)) F(dx) < +\infty \quad \text{on } \{Z_- < 1\}. \quad (\text{A.57})$$

(b) *Let f be a $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable and positive functional, and $\bar{\mu} := (1 - \tilde{Z}) \cdot \mu$. Then $\sqrt{(f-1)^2} I_{\llbracket \tau, +\infty \rrbracket} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if $\sqrt{(f-1)^2} I_{\{1 - Z_- \geq \delta\}} \star \bar{\mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$ for any $\delta > 0$.*

B Representation Results

Lemma B.1. *The following assertions hold.*

(a) *If $H^\mathbb{G}$ is a $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional, then there exist an $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional $H^\mathbb{F}$ such that*

$$H^\mathbb{G}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket} = H^\mathbb{F}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket}. \quad (\text{B.58})$$

(b) *If furthermore $H^\mathbb{G} > 0$ (respectively $H^\mathbb{G} \leq 1$), then we can choose $H^\mathbb{F} > 0$ (respectively $H^\mathbb{F} \leq 1$) such that*

$$H^\mathbb{G}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket} = H^\mathbb{F}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket}.$$

(c) *For any \mathbb{F} -stopping time, T , and any positive \mathcal{G}_T -measurable random variable $Y^\mathbb{G}$, there exist two positive \mathcal{F}_T -measurable random variables, $Y^{(1)}$ and $Y^{(2)}$, satisfying*

$$Y^\mathbb{G} I_{\{T \leq \tau\}} = Y^{(1)} I_{\{T < \tau\}} + Y^{(2)} I_{\{\tau = T\}}. \quad (\text{B.59})$$

Lemma B.2. *Suppose that τ is honest. Let $H^\mathbb{G}$ be an $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional. Then the following assertions hold.*

(a) *There exist two $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional $H^\mathbb{F}$ and $K^\mathbb{F}$ such that*

$$H^\mathbb{G}(\omega, t, x) = H^\mathbb{F}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket} + K^\mathbb{F}(\omega, t, x) I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{B.60})$$

(b) *If furthermore $H^\mathbb{G} > 0$ (respectively $H^\mathbb{G} \leq 1$), then we can choose $K^\mathbb{F} > 0$ (respectively $K^\mathbb{F} \leq 1$) such that*

$$H^\mathbb{G}(\omega, t, x) I_{\llbracket \tau, +\infty \rrbracket} = K^\mathbb{F}(\omega, t, x) I_{\llbracket \tau, +\infty \rrbracket}.$$

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